Lagrangian theory for perfect fluids

Sawa Manoff
Bulgarian Academy of Sciences
Institute for Nuclear Research and Nuclear Energy
Department of Theoretical Physics
Blvd. Tzarigradsko Chaussee 72
1784 Sofia - Bulgaria

e-mail address: smanov@inrne.bas.bg

Abstract

The theory of perfect fluids is reconsidered from the point of view of a covariant Lagrangian theory. It has been shown that the Euler-Lagrange equations for a perfect fluid could be found in spaces with affine connections and metrics from an unconstrained variational principle by the use of the method of Lagrangians with covariant derivatives (MLCD) and additional conditions for reparametrizations of the proper time of the mass elements (particles) of the perfect fluid. The last conditions are not related to the variational principle and are not considered as constraints used in the process of variations. The application of the whole structure of a Lagrangian theory with an appropriate choice of a Lagrangian invariant as the pressure of the fluid shows that the Euler-Lagrange equations with their corresponding energy-momentum tensors lead to Navier-Stokes' equation identical with the Euler equation for a perfect fluid in a space with one affine connection and metrics. The Navier-Stokes equations appear as higher order equations with respect to the Euler-Lagrange equations.

1 Introduction

1. Every classical field theory could be considered as a theory of a continuous media [1] $\div$ [4] with its kinematic characteristics. On the one side, a classical field theory is usually based on three essential structures:

(a) The Lagrangian density,

(b) The Euler-Lagrangian equations,

(c) The energy-momentum tensors.

The structures in (a) - (c) determine the scheme of a Lagrangian formalism for describing the properties and the evolution of a dynamic system. The finding out of the Euler-Lagrange equations and the energy-momentum tensors is related to a variational principle applied under certain mathematical conditions for the variables and their variations [5], [4].

On the other side, a theory of a fluid is based on

(a) A state equation

(b) A variational principle with a given Lagrangian density depending on the variables constructing the equation of state (rest mass density, entropy, enthalpy, temperature, pressure etc.) and the velocity of the points of the fluid.

2. Since a theory of a fluid is a special case of a theory of continuous media, it could be worked out by means of the same mathematical tools used in a classical field theory. Moreover, the Lagrangian invariant L in the construction of the Lagrangian density $\mathbf{L} := \sqrt{-d_g} \cdot L$, $d_g := det(g_{ij}) < 0$, could be interpreted as the pressure p of a physical system, described by its Lagrangian density $\mathbf{L}$. Here, the tensor $g = g_{ij} \cdot dx^i.dx^j$, $dx^i.dx^j := \frac{1}{2} \cdot (dx^i \otimes dx^j + dx^j \otimes dx^i)$ is the metric tensor field in a differentiable manifold M, $dimM = n$, considered as a model of the space ($n = 3$) or of the space-time ($n = 4$). The fact that the pressure p could be used as Lagrangian invariant $L = p$ has been observed by many authors [6] (see the citations there) but used only for very special forms of the pressure p. The application of the assumption $L := p$ could be extended to many classical field theories and to theories of different types of fluids [4].

3. Every Lagrangian density depends on a set of field variables. Part of them (or all of them) are considered as dynamic field variables, i.e. as variables which variations lead to the Euler-Lagrange equations or to energy-momentum tensors. If a field variable is not varied it is called a non-dynamic field variable. It depends on the problem to be solved which variables are considered as dynamic and which as non-dynamic field variables. Only in the case of finding out the energy-momentum tensors the Lie variation [5] of all field variables should be taken into account. If all dynamic field variables are independent to each other, there are no additional conditions between them. The variables are varied without any constraints and the variational principle used in that case is called *unconstrained variational principle*. If a part of the dynamic field variables are depending in some way to each other there are relations (constraints) between them. The variation of the variables have to include the constraints between them and the variational principle used in this case is called *constrained variational principle*. In many cases the constraints have no obvious physical interpretation or lead to additional problems. There are attempts for finding out conditions which could show us which field theory could be derived from an unconstrained variational principles and which cannot [6]. On this basis Schutz and Sorkin came to the conclusion that the theory of a perfect fluid could not be derived from an unconstrained variational principle. At that, they used the method of Lagrangians with partial derivatives (MLPD) [5]. The theory of a perfect fluid [related to the Euler equation in hydrodynamics (1727-1741)] is considered as the oldest of classical field theories [6] and it is worth to be investigated by the use of some of the modern mathematical tools with respect to its structures and elements.

4. In the present paper we have used the method of Lagrangians with covariant derivatives (MLCD) instead of the method of Lagrangians with partial derivatives (MLPD) to reconsider the problem of derivation of the theory of a perfect fluid from an unconstrained variational principle. It has been shown that

the Euler-Lagrange equations for a perfect fluid could be found in spaces with affine connections and metrics [3] from an unconstrained variational principle by the use of the method of Lagrangians with covariant derivatives (MLCD) and additional conditions for reparametrizations of the proper time of the mass elements (particles) of the perfect fluid. The last conditions are not related to the variational principle and are not considered as constraints used in the process of variations. The application of the whole structure of a Lagrangian theory with an appropriate choice of a Lagrangian invariant as the pressure of the fluid shows that the Euler-Lagrange equations with their corresponding energy-momentum tensors lead to Navier-Stokes' equations containing the Euler equation for a perfect fluid as a special case in a space with affine connections and metrics [7], [3]. The Navier-Stokes equation appear as higher order equations with respect to the Euler-Lagrange equations.

5. The structure of the paper is as follows. In Section 2 a brief recall of the method of Lagrangians with covariant derivatives (MLCD) is made and applied to a Lagrangian density depending on a metric, a scalar field, and on a contravariant non-isotropic (non-null) vector field. In Section 3 the invariant projections of the energy-momentum tensors are considered for the corresponding Lagrangian density.. In Section 4 the covariant divergencies of the energy-momentum tensors and their relations to Navier-Stokes' equations are given. In Section 5 the invariance of dynamic characteristics under changing the proper time of the trajectory of a mass element (particle) in a fluid is considered. In Section 6 the notion of perfect fluid and its mathematical models are considered. In Section 7 the method of Lagrangians with covariant derivatives is applied to a special type of a Lagrangian density related to a perfect fluid in special types of $(\overline{L}_n, g)$-spaces and in (L_n, g)-spaces and the corresponding Euler-Lagrange equations and energy-momentum tensors are found. In Section 8 the Navier-Stokes equation is considered in (L_n, g)-spaces. It is shown that this equation is identical with the Euler equation if considered for a perfect fluid with the given Lagrangian density in a (L_n, g)-space. The final Section 9 comprises some concluding remarks. The most considerations are given in details (even in full details) for these readers who are not familiar with the considered problems. If some details or abbreviations are not defined or introduced in the paper the reader is kindly asked to refer to [3] and [4] where all preliminary informations for reading this paper are given.

2 Method of Lagrangians with covariant derivatives (MLCD)

2.1 General remarks

The method of Lagrangians with covariant derivatives (MLCD) is considered in details in [5], [8], [4]. We will recall here only the main structures and properties of this method.

The *method of Lagrangians with covariant derivatives (MLCD)* is a Lagrangian formalism for tensor fields based on:

(a) A Lagrangian density $\mathbf{L}$ of type

$$\mathbf{L} = \sqrt{-d_g}.L(g_{ij}, g_{ij;k}, g_{ij;k;l}, V^A{}_B, V^A{}_{B;i}, V^A{}_{B;i;j}) \ , \tag{1}$$

where g_{ij} are the components of the covariant metric tensor field g, $_{;k}$ denotes covariant derivative with respect to the co-ordinate x^k (or with respect to the basic co-ordinate vector field ∂_k), $V^A{}_B$ are components of tensor fields $V \in \otimes^k{}_l(M)$ with finite rank, A and B are co-operative indices.

(b) The action S of a Lagrangian system described by means of the Lagrangian density $\mathbf{L}$

$$S = \int\limits_{V_n} \mathbf{L}.d^{(n)}x = \int\limits_{V_n} L.d\omega \ , \tag{2}$$

where V_n is a volume in the manifold M with $\dim M = n$; $d\omega = \sqrt{-d_g}.d^{(n)}x$ is the invariant volume element.

(c) The functional variation δS of the action S with the condition for the existence of an extremum

$$\delta S = \delta \int\limits_{V_n} \mathbf{L}.d^{(n)}x = \int\limits_{V_n} \delta\mathbf{L}.d^{(n)}x = 0 \ . \tag{3}$$

(d) The functional variation of the Lagrangian density $\mathbf{L}$ in the form

$$\delta\mathbf{L} = \frac{\partial\mathbf{L}}{\partial g_{ij}}.\delta g_{ij} + \frac{\partial\mathbf{L}}{\partial g_{ij;k}}.\delta(g_{ij;k}) + \frac{\partial\mathbf{L}}{\partial g_{ij;k;l}}.\delta(g_{ij;k;l}) +$$
$$+ \frac{\partial\mathbf{L}}{\partial V^A{}_B}.\delta V^A{}_B + \frac{\partial\mathbf{L}}{\partial V^A{}_{B;i}}.\delta(V^A{}_{B;i}) + \frac{\partial\mathbf{L}}{\partial V^A{}_{B;i;j}}\delta(V^A{}_{B;i;j}) \ . \tag{4}$$

(e) The variational operator δ obeying the commutation relations leading to commutation of δ with the covariant derivatives:

$$\begin{gathered} \delta(g_{ij;k}) = (\delta g_{ij})_{;k} \ , \quad \delta(g_{ij;k;l}) = (\delta g_{ij})_{;k;l} \ , \\ \delta(V^A{}_{B;i}) = (\delta V^A{}_B)_{;i} \ , \quad \delta(V^A{}_{B;i;j}) = (\delta V^A{}_B)_{;i;j} \ . \end{gathered} \tag{5}$$

(f) The Lie variation $\pounds_\xi S$ of the action S

$$\pounds_\xi S = \pounds_\xi \int\limits_{V_n} \mathbf{L}.d^{(n)}x = \int\limits_{V_n} \overline{\pounds}_\xi \mathbf{L}.d^{(n)}x \ . \tag{6}$$

(g) The Lie variation of the Lagrangian density $\mathbf{L}$ in the form

$$\overline{\pounds}_\xi \mathbf{L} \equiv \frac{\partial\mathbf{L}}{\partial g_{ij}}.\pounds_\xi g_{ij} + \frac{\partial\mathbf{L}}{\partial g_{ij;k}}.\pounds_\xi(g_{ij;k}) + \frac{\partial\mathbf{L}}{\partial g_{ij;k;l}}.\pounds_\xi(g_{ij;k;l}) +$$
$$+ \frac{\partial\mathbf{L}}{\partial V^A{}_B}.\pounds_\xi V^A{}_B + \frac{\partial\mathbf{L}}{\partial V^A{}_{B;i}}.\pounds_\xi(V^A{}_{B;i}) + \frac{\partial\mathbf{L}}{\partial V^A{}_{B;i;j}}.\pounds_\xi(V^A{}_{B;i;j}) \ . \tag{7}$$

Remark. In the MLCD [because of the commutation relations] *the affine connections* Γ *and* P, *and their corresponding curvature tensors are considered as non-dynamic field variables* ($\delta\Gamma^i_{jk} = 0$, $\delta P^i_{jk} = 0$, $\delta R^i{}_{jkl} = 0$, $\delta P^i{}_{jkl} = 0$). Therefore, in the MLCD variations of the components and their covariant derivatives of the covariant metric tensor g and the non-metric tensor fields V are considered for given (fixed) affine connections and for fixed and determined by them types of transports of the tensor fields. Of course, the question arises how the affine connections can be found if not by means of a Lagrangian formalism. The first simple answer is: the affine connections (or the equations for them as functions of the co-ordinates in M) can be found on the grounds of the method of Lagrangians with partial derivatives (MLPD) and then the components of the tensor fields (as functions of the co-ordinates in M) can be determined by means of MLCD. This answer could induce an other question: why two methods have to be applied when one is enough for finding out all equations for all dynamic field variables. There are at least two possible answers to this question: 1. The MLCD ensure the finding out equations for tensor fields (as dynamic field variables). These equations are (a) covariant (tensorial) equations and (b) form-invariant (gauge invariant) equations with respect to the affine connections. The affine connections, related to the type of the frame of reference [9] could be determined on the grounds of additional conditions and not exactly by means of a variational principle. 2. The MLPD can ensure the consideration of the affine connections as dynamic field variables and the finding out their field equations. It cannot give direct answer for the type (tensorial or non-tensorial) of the equations obtained for the tensor field variables and their corresponding energy-momentum quantities. The tensorial character of quantities and relations concerning the tensor fields variables has to be proved (which, in general, could be a matter of some difficulty) [10].

2.2 Method of Lagrangians with covariant derivatives applied to a scalar and to a vector field

2.2.1 Lagrangian density

1. The mathematical description of a state of a fluid is usually made by means of the velocity vector field u of the moving mass elements (particles) of the fluid and other two functions (for instance, the pressure p and the rest mass density ρ) [11]. If u, ρ, and p are given then the state of a fluid is determined. This fact gives rise to the idea a Lagrangian invariant $L = p$ depending on the velocity vector field u and on the rest mass density ρ to be used in the MLCD for finding out the Euler-Lagrange equations and their corresponding energy-momentum tensors in a space with affine connections and metrics [$(\overline{L}_n, g)$-space] (considered as a model of space or space-time).

2. Let a Lagrangian density of the type

$$\mathbf{L} := \sqrt{-d_g} \cdot p(g_{ij},\, \rho,\, \rho_{;i} \equiv \rho_{,i},\, u^i) \tag{8}$$

be given, where $d_g := det(g_{ij}) < 0$, $g_{ij} = g_{ji}(x^k)$ are the components of a covari-

ant metric g, ρ is an invariant function on M, $\rho \in C^r(M)$, $r \geq 2$, $\rho \in \otimes^0{}_0(M)$, $\rho_{;i} \equiv \rho_{,i} := \partial\rho/\partial x^i$, u^i are the components of a non-null (non-isotropic) vector field $u \in T(M)$ in a co-ordinate basis $\{\partial_i\} \subset T(M)$, $u = u^i \cdot \partial_i$, $g(u,u) := e :\neq 0$, $i, j = 1, ..., n$, $dimM = n$. The Lagrangian invariant $L := p$ could be interpreted as the pressure p of the fluid. The invariant function ρ could be interpreted as the rest mass density of the fluid corresponding to one of its energy-momentum tensors [5]. The contravariant non-isotropic (non-null) vector field u and its components u^i could be interpreted as the velocity vector field of the mass elements (particles) of the fluid.

By the use of the MLCD we can find the Euler-Lagrange equations and the energy-momentum tensors corresponding to $\mathbf{L}$. The Lagrangian density $\mathbf{L}$ is a degenerated Lagrangian density with respect to u^i and g_{ij}, i.e. $\mathbf{L}$ contains no derivatives of u and g_{ij} with respect to the co-ordinates x^i or to the basic vector fields $e_i \equiv \partial_i$.

2.2.2 Euler-Lagrange's equations

(a) Euler-Lagrange's equations for the rest mass density ρ

The ELE for the rest mass density ρ could be found in the form [5], [4]

$$\frac{\delta_\rho p}{\delta\rho} + P = 0 \quad , \tag{9}$$

where

$$\frac{\delta_\rho p}{\delta\rho} = \frac{\partial p}{\partial\rho} - (\frac{\partial p}{\partial\rho_{;i}})_{;i} \quad , \tag{10}$$

$$P = q_i \cdot \frac{\partial p}{\partial\rho_{;i}} \quad , \quad q_i = T_{ki}{}^k - \frac{1}{2} \cdot g^{\overline{kl}} \cdot g_{kl;i} + g^l_k \cdot g^k_{l;i} \quad . \tag{11}$$

Therefore, the ELE for ρ could be written as

$$\frac{\partial p}{\partial\rho} - (\frac{\partial p}{\partial\rho_{;i}})_{;i} + q_i \cdot \frac{\partial p}{\partial\rho_{;i}} = 0 \quad . \tag{12}$$

(b) Euler-Lagrange's equations for the vector field u

The Lagrangian density $\mathbf{L}$ is degenerated with respect to the vector field u, i.e. the ELEs could be written in the form [3], [4]

$$\frac{\partial p}{\partial u^i} = 0 \quad . \tag{13}$$

(c) Euler-Lagrange's equations for the metric tensor g

The Lagrangian density $\mathbf{L}$ is degenerated with respect to the components g_{ij} of the tensor field g. Therefore,

$$\frac{\delta_g p}{\delta g_{ij}} = \frac{\partial p}{\partial g_{ij}} \ ,$$

and we have the ELEs in the form

$$\frac{\partial p}{\partial g_{ij}} + \frac{1}{2} \cdot p \cdot g^{\overline{ij}} = 0 \quad . \tag{14}$$

If the ELEs for g_{ij} are fulfilled then the pressure p should obey the equation

$$\frac{\partial p}{\partial g_{ij}} \cdot g_{ij} + \frac{n}{2} \cdot p = 0 \qquad : \qquad p = -\frac{2}{n} \cdot \frac{\partial p}{\partial g_{ij}} \cdot g_{ij} \quad . \tag{15}$$

The last condition means that p should be a homogeneous function of g_{ij} of degree $m = -(2/n)$. If the pressure p does not have this property then the metric tensor g could not be considered as a dynamic variable, i.e. it should not be varied.

2.2.3 Energy-momentum tensors

In accordance to the general scheme of the MLCD we could find the explicit form of the corresponding to the pressure p energy-momentum tensors.

(a) Generalized canonical energy-momentum tensor

$$\overline{\theta}_i{}^j = \frac{\partial p}{\partial \rho_{;j}} \cdot \rho_{;i} - p \cdot g_i^j \quad . \tag{16}$$

(b) Symmetric energy-momentum tensor of Belinfante

$$_sT_i{}^j = -p \cdot g_i^j \quad . \tag{17}$$

(c) Variational energy-momentum tensor of Euler-Lagrange

$$\begin{aligned} \overline{Q}_i{}^j &= \frac{\delta_u p}{\delta u^k} \cdot g_i^k \cdot g_l^j \cdot u^l - \frac{\delta_g p}{\delta g_{kl}} \cdot g_{\overline{k}}^j \cdot g_{\overline{i}}^m \cdot g_{ml} - \frac{\delta_g p}{\delta g_{kl}} \cdot g_{\overline{l}}^j \cdot g_{\overline{i}}^m \cdot g_{km} = \\ &= \frac{\partial p}{\partial u^i} \cdot u^j - \frac{\partial p}{\partial g_{kl}} \cdot g_{\overline{k}}^j \cdot g_{\overline{i}}^m \cdot g_{ml} - \frac{\partial p}{\partial g_{kl}} \cdot g_{\overline{l}}^j \cdot g_{\overline{i}}^m \cdot g_{km} \quad . \end{aligned}$$

Because of $g_{\overline{i}k} = f^m{}_i \cdot g_{mk} = f^m{}_i \cdot g_{km} = g_{k\overline{i}}$, $g_{kl} = g_{lk}$, $g_{\overline{i}}^m \cdot g_{ml} = f^n{}_i \cdot g_n^m \cdot g_{ml} = f^n{}_i \cdot g_{nl} = g_{\overline{i}l}$ we have

$$\overline{Q}_i{}^j = \frac{\partial p}{\partial u^i} \cdot u^j - 2.\frac{\partial p}{\partial g_{kl}} \cdot g_{\overline{i}k} \cdot g_{\overline{l}}^j \quad . \tag{18}$$

3 Invariant projections of the energy-momentum tensors

Since the vector field u could be interpreted as the velocity of the mass elements (particles) of a fluid, we can project the energy-momentum tensors $\overline{\theta}_i{}^j$, $_sT_i{}^j$,

and $\overline{Q}_i{}^j$ by the use of the vector field u and its corresponding covariant and contravariant projective metrics h_u and h^u

$$
\begin{aligned}
h_u &= g - \frac{1}{e} \cdot g(u) \otimes g(u) \quad , \qquad h^u = \overline{g} - \frac{1}{e} \cdot u \otimes u \quad , \\
e &= g(u,u) :\neq 0 \quad , \quad \overline{g} = g^{ij} \cdot \partial_i . \partial_j \quad , \\
\partial_i . \partial_j &= \frac{1}{2} \cdot (\partial_i \otimes \partial_j + \partial_j \otimes \partial_i) \; , \\
g(u) &= g_{i\overline{j}} \cdot u^j \cdot dx^i = g_{ij} \cdot u^{\overline{j}} \cdot dx^i \quad .
\end{aligned} \tag{19}
$$

By the use of u, h_u, and h^u we can find the different invariant projections of $\overline{\theta}_i{}^j$, ${}_sT_i{}^j$, and $\overline{Q}_i{}^j$ which have physical interpretation.

3.1 Invariant projections of the generalized canonical energy-momentum tensor $\overline{\theta}$

1. Rest mass density ρ_θ.

$$
\begin{aligned}
\rho_\theta &= \frac{1}{e^2} \cdot \overline{\theta}_k{}^i \cdot u_{\overline{i}} \cdot u^{\overline{k}} = \\
&= \frac{1}{e^2} \cdot u_{\overline{i}} \cdot u^{\overline{k}} \cdot (\frac{\partial p}{\partial \rho_{;i}} \cdot \rho_{;k} - p \cdot g^i_k) = \\
&= \frac{1}{e^2} \cdot (\frac{\partial p}{\partial \rho_{;i}} \cdot u_{\overline{i}} \cdot u^{\overline{k}} \cdot \rho_{;k} - p \cdot u_{\overline{k}} \cdot u^{\overline{k}}) \; .
\end{aligned} \tag{20}
$$

2. Conductive momentum density ${}^\theta\overline{\pi}$.

$$
{}^\theta\overline{\pi}{}^i := \frac{1}{e} \cdot {}_k\overline{\theta}_l{}^k \cdot u_{\overline{k}} \cdot h^{\overline{l}i} = \frac{1}{e} \cdot \frac{\partial p}{\partial \rho_{;k}} \cdot \rho_{;l} \cdot u_{\overline{k}} \cdot h^{\overline{l}i} \quad . \tag{21}
$$

3. Conductive energy flux density $e \cdot {}^\theta\overline{s}$.

$$
{}^\theta\overline{s}{}^i := \frac{1}{e} \cdot h^{ij} \cdot g_{\overline{jk}} \cdot {}_k\overline{\theta}_l{}^k \cdot u^{\overline{l}} = \frac{1}{e} \cdot h^{ij} \cdot g_{\overline{jk}} \cdot \frac{\partial p}{\partial \rho_{;k}} \cdot \rho_{;l} \cdot u^{\overline{l}} \quad . \tag{22}
$$

$$
e \cdot {}^\theta\overline{s}{}^i = h^{ij} \cdot g_{\overline{jk}} \cdot \frac{\partial p}{\partial \rho_{;k}} \cdot \rho_{;l} \cdot u^{\overline{l}} \quad . \tag{23}
$$

4. Stress tensor density ${}^\theta\overline{S}$.

$$
{}^\theta\overline{S}{}^{ij} := h^{ik} \cdot g_{\overline{kl}} \cdot {}_k\overline{\theta}_m{}^l \cdot h^{\overline{m}j} = h^{ik} \cdot g_{\overline{kl}} \cdot \frac{\partial p}{\partial \rho_{;l}} \cdot \rho_{;m} \cdot h^{\overline{m}j} \quad . \tag{24}
$$

3.2 Invariant projections of the symmetric energy-momentum tensor of Belinfante

1. Rest mass density ρ_T.

$$
\rho_T := \frac{1}{e^2} \cdot {}_sT_k{}^i \cdot u_{\overline{i}} \cdot u^{\overline{k}} = - \frac{1}{e^2} \cdot p \cdot u_{\overline{k}} \cdot u^{\overline{k}} \quad . \tag{25}
$$

2. Conductive momentum density ${}^{T}\overline{\pi}$.

$$ {}^{T}\overline{\pi}^{\,i} := \frac{1}{e} \cdot T_{l}{}^{k} \cdot u_{\overline{k}} \cdot h^{\overline{l}i} = 0 \quad , \qquad T_{l}{}^{k} = 0 \quad . \tag{26} $$

3. Conductive energy flux density $e \cdot {}^{T}\overline{s}$.

$$ {}^{T}\overline{s}^{i} := \frac{1}{e} \cdot h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot T_{l}{}^{k} \cdot u^{\overline{l}} = 0 \quad , \quad T_{l}{}^{k} = 0 \quad . \tag{27} $$

4. Stress tensor density ${}^{T}\overline{S}$.

$$ {}^{T}\overline{S}^{ij} := h^{ik} \cdot g_{\overline{k}\overline{l}} \cdot T_{m}{}^{l} \cdot h^{\overline{m}j} = 0 \quad . \tag{28} $$

3.3 Invariant projections of the variational energy-momentum tensor of Euler-Lagrange

1. Rest mass density ρ_{Q}.

$$ \begin{aligned} \rho_{Q} &= -\frac{1}{e^{2}} \cdot \overline{Q}_{k}{}^{i} \cdot u_{\overline{i}} \cdot u^{\overline{k}} = \frac{2}{e^{2}} \cdot \frac{\partial p}{\partial g_{mn}} \cdot g_{\overline{k}m} \cdot g^{i}_{\overline{n}} \cdot u_{\overline{i}} \cdot u^{\overline{k}} - \\ &\quad -\frac{1}{e^{2}} \cdot \frac{\partial p}{\partial u^{k}} \cdot u^{i} \cdot u_{\overline{i}} \cdot u^{\overline{k}} \end{aligned} $$

$$ \rho_{Q} = \frac{2}{e^{2}} \cdot \frac{\partial p}{\partial g_{mn}} \cdot g_{\overline{k}m} \cdot u_{n} \cdot u^{\overline{k}} - -\frac{1}{e} \cdot \frac{\partial p}{\partial u^{k}} \cdot u^{\overline{k}} \quad . \tag{29} $$

2. Conductive momentum density ${}^{Q}\pi$.

$$ \begin{aligned} {}^{Q}\pi^{\,i} &= -\frac{1}{e} \cdot \overline{Q}_{l}{}^{k} \cdot u_{\overline{k}} \cdot h^{\overline{l}i} = \\ &= -\frac{1}{e} \cdot \left(\frac{\partial p}{\partial u^{l}} \cdot u^{k} \cdot u_{\overline{k}} \cdot h^{\overline{l}i} - 2 \cdot \frac{\partial p}{\partial g_{mn}} \cdot g_{\overline{l}m} \cdot g^{k}_{\overline{n}} \cdot u_{\overline{k}} \cdot h^{\overline{l}i}\right) = \\ &= \frac{2}{e} \cdot \frac{\partial p}{\partial g_{mn}} \cdot g_{\overline{l}m} \cdot u_{n} \cdot h^{\overline{l}i} - \frac{\partial p}{\partial u^{l}} \cdot h^{\overline{l}i} \quad . \end{aligned} \tag{30} $$

3. Conductive energy flux density $e \cdot {}^{Q}s$.

$$ \begin{aligned} {}^{Q}s^{i} &= -\frac{1}{e} \cdot h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot \overline{Q}_{l}{}^{k} \cdot u^{\overline{l}} = \\ &= -\frac{1}{e} \cdot h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot u^{\overline{l}} \cdot \left(\frac{\partial p}{\partial u^{l}} \cdot u^{k} - 2 \cdot \frac{\partial p}{\partial g_{mn}} \cdot g_{\overline{l}m} \cdot g^{k}_{\overline{n}}\right) = \\ &= \frac{2}{e} \cdot h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot u^{\overline{l}} \cdot g_{\overline{l}m} \cdot g^{k}_{\overline{n}} \cdot \frac{\partial p}{\partial g_{mn}} - \frac{1}{e} \cdot h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot u^{\overline{l}} \cdot u^{k} \cdot \frac{\partial p}{\partial u^{l}} \quad , \end{aligned} $$

$$ h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot u^{k} = 0 \quad , $$

$$ {}^{Q}s^{i} = \frac{2}{e} \cdot h^{ij} \cdot g_{\overline{j}\overline{n}} \cdot \frac{\partial p}{\partial g_{mn}} \cdot g_{m\overline{l}} \cdot u^{\overline{l}} \quad . \tag{31} $$

4. Stress tensor density ${}^{Q}S$.

$$\begin{aligned}
{}^{Q}S^{ij} &= -h^{ik} \cdot g_{\overline{kl}} \cdot \overline{Q}_m{}^l \cdot h^{\overline{m}j} = \\
&= -h^{ik} \cdot g_{\overline{kl}} \cdot (\frac{\partial p}{\partial u^m} \cdot u^l - 2 \cdot \frac{\partial p}{\partial g_{rs}} \cdot g_{\overline{m}r} \cdot g^{l}_{\overline{s}}) \cdot h^{\overline{m}j} = \\
&= -h^{ik} \cdot g_{\overline{kl}} \cdot u^l \cdot \frac{\partial p}{\partial u^m} \cdot h^{\overline{m}j} + 2 \cdot h^{ik} \cdot g_{\overline{kl}} \cdot \frac{\partial p}{\partial g_{rs}} \cdot g_{\overline{m}r} \cdot g^{l}_{\overline{s}} \cdot h^{\overline{m}j} \quad ,
\end{aligned}$$

$${}^{Q}S^{ij} = 2 \cdot h^{ik} \cdot g_{\overline{kl}} \cdot \frac{\partial p}{\partial g_{nl}} \cdot g_{\overline{mn}} \cdot h^{\overline{m}j} \quad . \tag{32}$$

3.4 Noether's identities

From the second covariant Noether identity

$$\overline{\theta} - {}_{s}T \equiv Q \tag{33}$$

we obtain the identities:

1. $\rho_\theta \equiv \rho_T - \rho_Q$:

$$\frac{\partial p}{\partial \rho_{;i}} \cdot u_{\overline{i}} \cdot u^{\overline{k}} \cdot \rho_{;k} \equiv e \cdot \frac{\partial p}{\partial u^k} \cdot u^{\overline{k}} - 2 \cdot \frac{\partial p}{\partial g_{mn}} \cdot g_{\overline{k}m} \cdot u_n \cdot u^{\overline{k}} \quad . \tag{34}$$

2. ${}^{\theta}\overline{\pi} \equiv {}^{T}\overline{\pi} - {}^{Q}\pi$:

$$\frac{\partial p}{\partial \rho_{;k}} \cdot \rho_{;l} \cdot u_{\overline{k}} \cdot h^{\overline{l}i} \equiv e \cdot \frac{\partial p}{\partial u^l} \cdot h^{\overline{l}i} - 2 \cdot \frac{\partial p}{\partial g_{mn}} \cdot g_{\overline{l}m} \cdot u_n \cdot h^{\overline{l}i} \quad . \tag{35}$$

3. ${}^{\theta}\overline{s} \equiv {}^{T}\overline{s} - {}^{Q}s$:

$$h^{ij} \cdot g_{\overline{jk}} \cdot \frac{\partial p}{\partial \rho_{;k}} \cdot \rho_{;l} \cdot u^{\overline{l}} \equiv -2 \cdot h^{ij} \cdot g_{\overline{j}k} \cdot \frac{\partial p}{\partial g_{km}} \cdot g_{m\overline{l}} \cdot u^{\overline{l}} \quad . \tag{36}$$

4. ${}^{\theta}\overline{S} \equiv {}^{T}\overline{S} - {}^{Q}S$:

$$h^{ik} \cdot g_{\overline{kl}} \cdot \frac{\partial p}{\partial \rho_{;l}} \cdot \rho_{;m} \cdot h^{\overline{m}j} \equiv 2 \cdot h^{ik} \cdot g_{\overline{k}l} \cdot \frac{\partial p}{\partial g_{nl}} \cdot g_{\overline{mn}} \cdot h^{\overline{m}j} \quad . \tag{37}$$

3.5 Explicit form of the energy-momentum tensors

1. Generalized canonical energy-momentum tensor $\overline{\theta}$.

$$\overline{\theta} = (\rho_\theta + \frac{1}{e} \cdot k \cdot L) \cdot u \otimes g(u) - p \cdot Kr + u \otimes g({}^{\theta}\overline{\pi}) + {}^{\theta}\overline{s} \otimes g(u) + ({}^{\theta}\overline{S})g \quad . \tag{38}$$

2. Symmetric energy-momentum tensor of Belinfante ${}_{s}T$.

$${}_{s}T = -p \cdot Kr \quad . \tag{39}$$

3. Variational energy-momentum tensor of Euler-Lagrange Q.

$$Q = -\rho_Q \cdot u \otimes g(u) - u \otimes g({}^Q\pi) - {}^Q s \otimes g(u) - ({}^Q S)g \quad . \tag{40}$$

The form of the symmetric energy-momentum tensor of Belinfante shows that a fluid, described by a Lagrangian density of the type $\mathbf{L} = \sqrt{-d_g} \cdot p(g_{ij}, \rho, \rho_{;i}, u^i)$, could be considered as a fluid in equilibrium state [12]. Such an equilibrium state could be related to the Euler equation of a static state of a fluid.

4 Covariant divergencies of the energy-momentum tensors and Navier-Stokes' equations

4.1 Covariant divergency

The covariant divergency of the energy-momentum tensors could be written in the forms respectively [2], [4]

1. Covariant divergency of the generalized canonical energy-momentum tensor.

$$\begin{aligned} \overline{g}(\delta\overline{\theta}) &= (\rho_\theta + \frac{1}{e} \cdot k \cdot p) \cdot a + \\ &+ [u(\rho_\theta + \frac{1}{e} \cdot k \cdot p) + (\rho_\theta + \frac{1}{e} \cdot k \cdot p) \cdot \delta u + \delta^\theta \overline{s}] \cdot u - \\ &- \overline{g}(Krp) - p \cdot \overline{g}(\delta Kr) + \delta u \cdot {}^\theta\overline{\pi} + \nabla_u {}^\theta\overline{\pi} + \nabla_{{}^\theta\overline{s}} u + \\ &+ (\rho_\theta + \frac{1}{e} \cdot k \cdot p) \cdot \overline{g}(\nabla_u g)(u) + \overline{g}(\nabla_u g)({}^\theta\overline{\pi}) + \overline{g}(\nabla_{{}^\theta\overline{s}} g)(u) + \\ &+ \overline{g}(\delta(({}^\theta\overline{S})g)) \quad . \end{aligned} \tag{41}$$

2. Covariant divergency of the symmetric energy-momentum tensor of Belinfante.

$$\begin{aligned} \delta\, {}_s T &= -\delta(p \cdot Kr) = -(Krp + p \cdot \delta Kr) \quad , \\ \overline{g}(\delta_s T) &= -\overline{g}(Krp) - p \cdot \overline{g}(\delta Kr) \quad . \end{aligned} \tag{42}$$

3. Covariant divergency of the variational energy-momentum tensor of Euler-Lagrange.

$$\begin{aligned} \overline{g}(\delta Q) &= -\rho_Q \cdot a - (u\rho_Q + \rho_Q \cdot \delta u + \delta\, {}^Q s) \cdot u - \\ &- \delta u \cdot {}^Q\pi - \nabla_u {}^Q\pi - \nabla_{{}^Q s} u - \\ &- \rho_Q \cdot \overline{g}(\nabla_u g)(u) - \overline{g}(\nabla_u g)({}^Q\pi) - \overline{g}(\nabla_{{}^Q s} g)(u) - \overline{g}(\delta(({}^Q S)g)) \end{aligned} \tag{43}$$

4.2 Navier-Stokes' equations

The Navier-Stokes equations follow from the second Navier-Stokes identity [4]

$$h_u[\overline{g}(F)] + h_u[\overline{g}(\delta\theta)] \equiv 0 \tag{44}$$

in the form

$$h_u[\overline{g}(\delta\theta)] = 0 \quad . \tag{45}$$

In its explicit form the Navier-Stokes equation could be written as

$$\begin{aligned}
&(\rho_\theta + \frac{1}{e} \cdot k \cdot p) \cdot h_u(a) - h_u[\overline{g}(Krp)] - p \cdot h_u[\overline{g}(\delta Kr)] + \\
&+ \delta u \cdot h_u({}^\theta\overline{\pi}) + h_u(\nabla_u {}^\theta\overline{\pi}) + h_u(\nabla_{{}^\theta\overline{s}} u) + \\
&+ (\rho_\theta + \frac{1}{e} \cdot k \cdot p) \cdot h_u[\overline{g}(\nabla_u g)(u)] + h_u[\overline{g}(\nabla_u g)({}^\theta\overline{\pi})] + \\
&\quad + h_u[\overline{g}(\nabla_{{}^\theta\overline{s}} g)(u)] + h_u[\overline{g}(\delta(({}^\theta\overline{S})g))] = 0 \quad .
\end{aligned} \tag{46}$$

If we write the explicit form of the expressions

$$h_u(a) = g(a) - \frac{1}{e} \cdot g(u, a) \cdot g(u) \quad , \tag{47}$$

$$\begin{aligned}
h_u[\overline{g}(Krp)] &= g[\overline{g}(Krp)] - \frac{1}{e} \cdot g(u, \overline{g}(Krp)) \cdot g(u) = \\
&= Krp - \frac{1}{e} \cdot g(u, \overline{g}(Krp)) \cdot g(u)
\end{aligned} \tag{48}$$

in the Navier-Stokes equation we obtain it in the form

$$\begin{aligned}
&(\rho_\theta + \frac{1}{e} \cdot k \cdot p) \cdot g(a) - \frac{1}{e} \cdot (\rho_\theta + \frac{1}{e} \cdot k \cdot p) \cdot g(u, a) \cdot g(u) - \\
&- Krp + \frac{1}{e} \cdot g(u, \overline{g}(Krp)) \cdot g(u) - p \cdot h_u[\overline{g}(\delta Kr)] + \\
&+ \delta u \cdot h_u({}^\theta\overline{\pi}) + h_u(\nabla_u {}^\theta\overline{\pi}) + h_u(\nabla_{{}^\theta\overline{s}} u) + \\
&+ (\rho_\theta + \frac{1}{e} \cdot k \cdot p) \cdot h_u[\overline{g}(\nabla_u g)(u)] + h_u[\overline{g}(\nabla_u g)({}^\theta\overline{\pi})] + \\
&\quad + h_u[\overline{g}(\nabla_{{}^\theta\overline{s}} g)(u)] + h_u[\overline{g}(\delta(({}^\theta\overline{S})g))] = 0 \quad .
\end{aligned} \tag{49}$$

After contracting the last (previous) equation with $\overline{g}$ and introducing the abbreviation

$$\overline{\rho} := (\rho_\theta + \frac{1}{e} \cdot k \cdot p) \tag{50}$$

we find the Navier-Stokes equation in the form

$$\begin{aligned}
&\overline{\rho} \cdot a - \overline{g}(Krp) - \frac{1}{e} \cdot [\overline{\rho} \cdot g(u, a) - g(u, \overline{g}(Krp))] \cdot u - \\
&- p \cdot \overline{g}(h_u)[\overline{g}(\delta Kr)] + \delta u \cdot \overline{g}(h_u)({}^\theta\overline{\pi}) + \overline{g}(h_u)(\nabla_u {}^\theta\overline{\pi}) + \overline{g}(h_u)(\nabla_{{}^\theta\overline{s}} u) + \\
&+ \overline{\rho} \cdot \overline{g}(h_u)[\overline{g}(\nabla_u g)(u)] + \overline{g}(h_u)[\overline{g}(\nabla_u g)({}^\theta\overline{\pi})] + \\
&\quad + \overline{g}(h_u)[\overline{g}(\nabla_{{}^\theta\overline{s}} g)(u)] + \overline{g}(h_u)[\overline{g}(\delta(({}^\theta\overline{S})g))] = 0 \quad .
\end{aligned} \tag{51}$$

Special case: $(\overline{L}_n, g)$-spaces. ${}^{\theta}\overline{\pi} := 0$, ${}^{\theta}\overline{s} := 0$, ${}^{\theta}\overline{S} := 0$.

$$\overline{\rho} \cdot a - \overline{g}(Krp) - \frac{1}{e} \cdot [\overline{\rho} \cdot g(u,a) - g(u, \overline{g}(Krp))] \cdot u -$$

$$-p \cdot \overline{g}(h_u)[\overline{g}(\delta Kr)] + \overline{\rho} \cdot \overline{g}(h_u)[\overline{g}(\nabla_u g)(u)] = 0 \ . \tag{52}$$

Since $\overline{g}(h_u)\overline{g} = h^u$, we have

$$\overline{\rho} \cdot a - \overline{g}(Krp) - \frac{1}{e} \cdot [\overline{\rho} \cdot g(u,a) - g(u, \overline{g}(Krp))] \cdot u -$$

$$-p \cdot h^u(\delta Kr) + \overline{\rho} \cdot h^u[(\nabla_u g)(u)] = 0 \ . \tag{53}$$

Special case: $\overline{U}_n$- and $\overline{V}_n$-spaces. $\nabla_{\xi} g = 0$ for $\forall \xi \in T(M)$, ${}^{\theta}\overline{\pi} := 0$, ${}^{\theta}\overline{s} := 0$, ${}^{\theta}\overline{S} := 0$.

$$\overline{\rho} \cdot a - \overline{g}(Krp) - \frac{1}{e} \cdot [\overline{\rho} \cdot g(u,a) - g(u, \overline{g}(Krp))] \cdot u -$$

$$-p \cdot h^u(\delta Kr) = 0 \ . \tag{54}$$

Special case: U_n- and V_n-spaces. $\nabla_{\xi} g = 0$ for $\forall \xi \in T(M)$, ${}^{\theta}\overline{\pi} := 0$, ${}^{\theta}\overline{s} := 0$, ${}^{\theta}\overline{S} := 0$.

Since $\nabla_u g = 0$ and $\delta Kr = 0$ we have the Navier-Stokes equation in the form

$$\overline{\rho} \cdot a - \overline{g}(Krp) - \frac{1}{e} \cdot [\overline{\rho} \cdot g(u,a) - g(u, \overline{g}(Krp))] \cdot u = 0 \ . \tag{55}$$

The contravariant non-null (non-isotropic) vector field u is interpreted as the velocity vector field of the moving mass elements (material points, particles) in the fluid. The vector field a is interpreted as the acceleration of the same mass elements of the fluid. If we write the Navier-Stokes equation in a basis (in a co-ordinate or non-co-ordinate basis) we obtain its form

$$\overline{\rho} \cdot a^i - g^{\overline{ij}} \cdot p_{,j} - \frac{1}{e} \cdot [\overline{\rho} \cdot g_{\overline{kl}} \cdot u^k \cdot a^l - u^{\overline{l}} \cdot p_{,l}] \cdot u^i = 0 \quad . \tag{56}$$

On the other side,

$$\begin{aligned} g(u,a) &= \frac{1}{2} \cdot [ue - (\nabla_u g)(u,u)] \quad , \qquad a = \nabla_u u \quad , \\ \nabla_u e &= \nabla_u [g(u,u)] = ue = (\nabla_u g)(u,u) + 2 \cdot g(a,u) \quad . \end{aligned}$$

4.2.1 Navier-Stokes' equation in U_n- and V_n-spaces

In a U_n- or V_n-space we obtain the Navier-Stokes equation in the form

$$\overline{\rho} \cdot a^i - g^{ij} \cdot p_{,j} - \frac{1}{e} \cdot [\frac{1}{2} \cdot \overline{\rho} \cdot e_{,j} \cdot u^j - u^j \cdot p_{,j}] \cdot u^i = 0 \quad . \tag{57}$$

If the last term of the type of $f \cdot u$ vanishes then the expression takes the form of the *Euler equation*

$$\overline{\rho} \cdot a^i = g^{ij} \cdot p_{,j} \quad . \tag{58}$$

The vanishing of the term $f \cdot u$ with

$$f := \frac{1}{e} \cdot [\frac{1}{2} \cdot \overline{\rho} \cdot e_{,j} \cdot u^j - u^j \cdot p_{,j}] \tag{59}$$

could be achieved in two different ways:

(a) By additional conditions (constraints) on the vector field u and on the pressure p. If $e :=$ const. $\neq 0$ and $\frac{dp}{d\tau} := 0$ (for $u = \frac{d}{d\tau}$) then $f = 0$. These additional conditions should be included as constraints on the Lagrangian invariant p or introduced by definition of the pressure p and the velocity u. The condition $e :=$ const. $\neq 0$ is usually assumed in the relativistic mechanics in V_4-spaces. $\frac{dp}{d\tau} = 0$ should be in some way included in the Lagrangian formalism for p. This is the reason for the statement that the Euler equation could not be found on the basis of an unconstrained variational principle.

(b) By changing the proper time τ as the parameter of the trajectory (world line) $x^i(\tau)$ of a mass element. This change is not related to a variational principle and could be performed after all steps related to the Lagrangian formalism. The change of the proper time τ is related to the canonical form of an auto-parallel (or geodesic in a V_n-space) equation.

Let us now consider the consequences from the changing the proper time τ in an equation of motion of a mass element (particle) in the Newtonian and relativistic dynamics.

5 Invariance of dynamic characteristics under changing the proper time of the trajectory of a mass element (particle)

1. One interesting question is related to the invariance of physical laws under time's transformations. The general covariance of a dynamic law under coordinate transformations (diffeomorphisms of the differentiable manifolds considered as models of space or space-time) is well known and used in different physical theories. Let us now consider how a dynamic law changes under a change of the time's parameter in it. A good example for the behavior of an equation under changing the time's parameter in it is the auto-parallel equation in spaces with affine connections and metrics. It is well known that the change of the proper time is related to the canonical or non-canonical form of an auto–parallel equation [13], [3], [4]. Nevertheless, the physical interpretation of a change of the proper time is not considered in details. What is the physical interpretation of the change of the time's parameter in Newtonian and relativistic physics?

5.1 Transformation of the time's parameter in the second law of the Newtonian mechanics

The second law in the Newton mechanics could be written in the form

$$\nabla_u p = F \quad , \qquad u = \frac{d}{dt} \quad , \quad p = \rho \cdot u \quad , \quad \rho = \rho(x^k, u, t) \;\; , \;\; F = F(x^k, u, t) \quad . \tag{60}$$

If the parameter t interpreted as time's parameter is changed with a new parameter τ with

$$t = t(\tau), \quad \tau = \tau(t), \quad t, \tau \in C^r(M) \;\; , \quad r \geq 2 \;\; , \quad dimM = n \quad (n = 3) , \tag{61}$$

$$\frac{dt}{d\tau} = \alpha(x^k) = \alpha(x^k(\tau)) = \alpha(x^k(t)) \;\; , \; \overline{u} := \frac{d}{d\tau} = \alpha \cdot u = \frac{dt}{d\tau} \cdot \frac{d}{dt} \quad , \tag{62}$$

$$F = F(x^k, u, t) = F(x^k \; , \; \frac{1}{\alpha} \cdot \overline{u} \; , \; t(\tau)) \quad . \tag{63}$$

Then the second law, written by the use of the new time's parameter τ will have the form

$$\nabla_u(\rho \cdot u) = \frac{1}{\alpha^2} \cdot \nabla_{\overline{u}}(\rho \cdot \overline{u}) - \frac{1}{\alpha^2} \cdot \rho \cdot (u\alpha) \cdot \overline{u} = F(x^k \; , \; \frac{1}{\alpha} \cdot \overline{u} \; , \; t(\tau)) \quad . \tag{64}$$

Therefore,

$$\nabla_u p = \frac{1}{\alpha^2} \cdot \nabla_{\overline{u}}\overline{p} - \frac{1}{\alpha^2} \cdot \rho \cdot (u\alpha) \cdot \overline{u} = F \quad , \tag{65}$$

$$\nabla_{\overline{u}}\overline{p} = \rho \cdot (u\alpha) \cdot \overline{u} + \alpha^2 \cdot F = \overline{F} + \overline{f} \cdot \overline{u} \quad , \tag{66}$$

$$\nabla_{\overline{u}}\overline{p} = \overline{f} \cdot \overline{u} + \overline{F} \quad . \tag{67}$$

where

$$\overline{p} = \rho \cdot \overline{u} \quad , \qquad \overline{f} = \rho \cdot (u\alpha) \quad , \quad \overline{F} = \alpha^2 \cdot F \quad .$$

The change of the time's parameter t in the Newton second law generates in general an additional force proportional to the velocity u (or $\overline{u}$). A force of this type (proportional to the velocity) is usually interpreted as a *friction force*. Only under a transformation $t = t(\tau)$ for which

$$u\alpha = \frac{1}{\alpha} \cdot \overline{u}\alpha = \frac{1}{\alpha} \cdot \frac{d}{d\tau}(\frac{dt}{d\tau}) = \frac{1}{\alpha} \cdot \frac{d^2 t}{d\tau^2} = 0 \quad , \quad \alpha \neq 0 \quad , \tag{68}$$

the second Newton law remains invariant up to the constant function $(1/\alpha^2)$

$$\nabla_u p = \frac{1}{\alpha^2} \cdot \nabla_{\overline{u}}\overline{p} = F \quad . \tag{69}$$

The condition

$$\frac{d^2 t}{d\tau^2} = 0 \tag{70}$$

leading to the conditions

$$\frac{dt}{d\tau} = a_0 = \text{ const.}, \qquad t = a_0 \cdot \tau + b_0 \quad , \qquad b_0 = \text{ const.}, \tag{71}$$

shows that only under linear transformations of the time $t = t(\tau)$ the Newton second law is form invariant

$$\nabla_u p = F \quad , \quad \nabla_{\overline{u}} \overline{p} = \overline{F} \quad . \tag{72}$$

If a mass element (particle) with rest mass density ρ is a free test particle $(F = 0)$ moving under condition

$$\nabla_u p = 0 \quad , \tag{73}$$

then the change of the parameter t generates a friction force $\alpha \cdot \overline{f} \cdot u$ and the considered motion of the particle is no more a free motion but a motion under a friction force proportional to its velocity. Therefore, we can generate or remove friction forces in Newtonian mechanics by performing an appropriate transformation of the time's parameter t.

Remark. Since $\overline{u} = \alpha \cdot u$, the Lie derivative of $\overline{u}$ along u is $\pounds_u \overline{u} = [u, \overline{u}] = (u\alpha) \cdot u$. The dragging of $\overline{u}$ along u should not change $\overline{u}$ if we consider a free motion of a particle. This is so, because of the equation of motion $\nabla_u u = 0$ leading to $\nabla_{\overline{u}} u = 0$ and to the condition $\nabla_{\overline{u}} \overline{u} = 0$. Therefore, the Lie derivative $\pounds_u \overline{u} = \nabla_u \overline{u} - \nabla_{\overline{u}} u$ in a E_n- or V_n-space $(n = 3, 4)$ should vanish. The Lie derivative $\pounds_u \overline{u} = -\pounds_{\overline{u}} u$ vanishes if and only if $u\alpha = 0$ or $\overline{u}\alpha = 0$. This means that the function α should be a constant function, different from zero, on trajectories with tangent vectors u and $\overline{u}$. Under this condition $(u\alpha = 0)$ the momentum of a free moving particle is an invariant quantity $\nabla_u p = \nabla_{\overline{u}} \overline{p}$ under changing the time's parameter of the trajectory of the particle.

5.2 Transformation of the time's parameter for a free moving spinless test particle in spaces with affine connections and metrics

1. If we consider the auto-parallel equation as the equation for description of the motion of a free spinless mass element (test particle) with constant rest mass density we should distinguish its two forms [13], [14]

(a) Canonical form

$$\nabla_u u = 0 \quad , \tag{74}$$

(b) Non-canonical form

$$\nabla_u u = f \cdot u \quad . \tag{75}$$

2. The transition from the canonical to the non-canonical form could be realized to a transformation of the proper time (parameter) of the trajectory of the particle.

If $u := d/d\tau$ and $\overline{u} := (1/\alpha) \cdot u := d/d\overline{\tau}$ then

$$\begin{aligned} \nabla_u u &= \nabla_{\alpha \cdot \overline{u}} (\alpha \cdot \overline{u}) = \alpha \cdot [(\overline{u}\alpha) \cdot \overline{u} + \alpha \cdot \nabla_{\overline{u}} \overline{u}] = \\ &= \alpha^2 \cdot \nabla_{\overline{u}} \overline{u} + (\overline{u}\alpha) \cdot u = \alpha^2 \cdot \overline{a} + f \cdot u \quad , \end{aligned} \tag{76}$$

$$a = f \cdot u + \alpha^2 \cdot \overline{a} \quad , \quad f = \overline{u}\alpha \quad , \quad \overline{a} = \nabla_{\overline{u}} \overline{u} \quad . \tag{77}$$

The acceleration a after changing the parameter τ is expressed by two terms: $f \cdot u$ (proportional to u) and $\alpha^2 \cdot \overline{a}$ proportional to the acceleration $\overline{a}$ defined by the new proper time $\overline{\tau}$. The term $f \cdot u$ defines a friction acceleration.

3. If the auto-parallel equation is given in its non-canonical form as $\nabla_u u = \overline{f} \cdot u$ with $u = d/d\tau$ then we can change the parameter τ with a new parameter $\overline{\tau} := \overline{\tau}(\tau)$, $\tau = \tau(\overline{\tau})$, so that

$$\overline{u} = \frac{d}{d\overline{\tau}} = \frac{1}{\overline{\alpha}} \cdot u = \frac{d\tau}{d\overline{\tau}} \cdot \frac{d}{d\tau} \quad , \qquad \frac{d\tau}{d\overline{\tau}} = \frac{1}{\overline{\alpha}} \quad . \tag{78}$$

The auto-parallel equation $\nabla_u u = \overline{f} \cdot u$ could be written in the forms

$$\nabla_u u = \nabla_{\overline{\alpha} \cdot \overline{u}}(\overline{\alpha} \cdot \overline{u}) = \overline{f} \cdot \overline{\alpha} \cdot \overline{u} \quad , \tag{79}$$

$$[\overline{\alpha} \cdot (\overline{u}\overline{\alpha}) - \overline{f} \cdot \overline{\alpha}] \cdot \overline{u} + \overline{\alpha}^2 \cdot \nabla_{\overline{u}} \overline{u} = 0 \quad . \tag{80}$$

The function $\overline{\alpha}$ could be chosen in such a way that the condition

$$\overline{\alpha} \cdot (\overline{u}\overline{\alpha}) - \overline{f} \cdot \overline{\alpha} = 0 \tag{81}$$

could be fulfilled. Since $\overline{\alpha} :\neq 0$, we have

$$\begin{aligned} \overline{u}\overline{\alpha} - \overline{f} &= 0 : \overline{\alpha} = \overline{\alpha}_0 + \int \overline{f} \cdot d\overline{\tau} \quad , \quad \overline{\alpha}_0 = \text{ const.}, \\ \overline{f} &= \overline{f}(\tau) = \overline{f}(\tau(\overline{\tau})) = \overline{f}(\overline{\tau}) \ , \end{aligned} \tag{82}$$

or

$$\overline{\alpha} = \overline{\alpha}_0 \cdot exp[\int \overline{f}(\tau) \cdot d\tau] \quad , \quad \overline{\alpha}_0 = \text{ const. } . \tag{83}$$

If $\overline{\alpha}$ fulfills the previous condition then $\nabla_{\overline{u}} \overline{u} = 0$ and the auto-parallel equation is represented in its canonical form.

Therefore, an acceleration of the type of a friction acceleration (proportional to the velocity vector field of a mass element) could be generated or removed by the use of transformation of the proper time of the mass element. This fact could be used for removing terms proportional to the velocity in the structures of the energy-momentum tensors and in the structure of Navier-Stokes' equation.

4. Let as now assume that the acceleration a of a mass element with rest mass density ρ fulfills the equation

$$\nabla_u u = a = f \cdot u \quad , \qquad f \in C^r(M) \ , \quad r \geq 2 \quad .$$

Then the change of the momentum $p = \rho \cdot u$ of a mass element moving with the above acceleration could be written in the form

$$\begin{aligned} \nabla_u p &= \nabla_u(\rho \cdot u) = (u\rho) \cdot u + \rho \cdot a = \\ &= (u\rho) \cdot u + \rho \cdot f \cdot u = (u\rho + f \cdot \rho) \cdot u \quad . \end{aligned} \tag{84}$$

If the momentum p does not change under the acceleration $a = f \cdot u$, i.e. if

$$\nabla_u p = \nabla_u (\rho \cdot u) = 0 \tag{85}$$

then

$$u\rho + f \cdot \rho = 0 \quad , \qquad u \neq 0 \quad , \quad \rho \neq 0 \quad , \tag{86}$$

and the rest mass density ρ should obey the equation

$$\frac{u\rho}{\rho} = -f \quad , \quad u(log\rho) = -f \quad , \quad d(log\rho) = -f \cdot d\tau \quad ,$$

and therefore,

$$\rho = \rho_0 \cdot exp[-\int f(\tau) \cdot d\tau] \quad , \quad \rho_0 = \text{ const.} \tag{87}$$

Special case: If $f := \lambda :=$ const. $:\geq 0$, then

$$\rho = \rho_0 \cdot e^{-\lambda \cdot \tau} \quad . \tag{88}$$

5.2.1 Law of self-decay of a system of n-particles in a $(\overline{L}_n, g)$-space

Let us now assume as above that the rest mass density ρ of a mass element with equation of motion of the type $\nabla_u u = f \cdot u$ is a composition of the constant rest mass density $\overline{\rho}_0$ of n particles, i.e.

$$\rho := n \cdot \overline{\rho}_0 \quad , \qquad \overline{\rho}_0 = \text{ const.} \tag{89}$$

Then

$$\begin{aligned} \nabla_u(\rho \cdot u) &= \nabla_u(n \cdot \ \overline{\rho}_0 \cdot u) = \ \overline{\rho}_0 \cdot (un) \cdot u + \ \overline{\rho}_0 \cdot n \cdot \nabla_u u = \\ &= (un) \cdot \ \overline{\rho}_0 \cdot u + \ \overline{\rho}_0 \cdot n \cdot f \cdot u = \\ &= [un + n \cdot f] \cdot \overline{\rho}_0 \cdot u \ . \end{aligned} \tag{90}$$

If $\nabla_u p = \nabla_u(\overline{\rho}_0 \cdot n \cdot u) = 0$, i.e. if the momentum of the mass element with momentum p and with rest mass density $\rho = n \cdot \overline{\rho}_0$ does not change along its trajectory (world line), then

$$un + n \cdot f = 0 \tag{91}$$

and

$$n = n_0 \cdot exp[-\int_{\tau_0}^{\tau} f \cdot d\tau] \quad , \qquad n_0 = \text{ const.} \tag{92}$$

The number n is the number of the particles in the mass element ρ at the time τ. n_0 is the number of the particles in the mass element ρ at the time τ_0.

$$n(\tau) = n(\tau_0) \cdot e^{[-\int_{\tau_0}^{\tau} f \cdot d\tau]} \quad . \tag{93}$$

Special case: $f := \lambda :=$ const. ≥ 0.

$$n(\tau) = n(\tau_0) \cdot e^{-\lambda \cdot \tau} \quad . \tag{94}$$

The last expression is exactly the law of self-decay of a system of n particles in the time τ. Therefore, if a mass element with rest mass density ρ (consisting of n particles with rest mass density $\rho_0 = const.$) does not change its momentum density $p = \rho \cdot u$ along its world line with tangent vector u under an acceleration of the friction type $a = f \cdot u$ with $f = \lambda =$ const. then the number n of the particles in the mass element should obey the law of self-decay

$$n(\tau) = n(\tau_0) \cdot e^{-\lambda \cdot \tau} \quad , \qquad n_0 = \text{ const. }, \quad \lambda = \text{ const.} \tag{95}$$

It follows that from the one side, accelerations of the type $a = f \cdot u$ could be brought to zero by a corresponding transformation of the proper time τ and on the other side, it could be important for the description of a self-decay law on a classical level.

The consideration of the law of self-decay on an elementary mechanical level shows that the reparametrization of the proper time could have important for the classical physics corollaries. The motion under friction acceleration and preservation of the momentum of a mass element despite of this friction acceleration (which can be removed by a proper time reparametrization) could lead on the one side, to a self-decay of a mass element (emission of particles), but on the other side, it could lead to absorption of particles from the vicinity of a mass element ($f := \overline{\lambda} :=$ const. ≤ 0). The number of the emitted or absorbed particles n could also depend on the proper time τ (related to the mass element ρ) in a more complicated manner than on the exponential law.

5.3 Transformation of the proper time of a mass element in the Navier-Stokes equation in $(\overline{L}_n, g)$-spaces

1. Now we can again consider the Navier-Stokes equation in the special case when ${}^{\theta}\overline{\pi} := 0$, ${}^{\theta}\overline{s} := 0$, and ${}^{\theta}\overline{S} := 0$. It could be written in the form

$$\overline{\rho} \cdot a - \overline{g}(Krp) - \overline{f} \cdot u - p \cdot h^u(\delta Kr) + \overline{\rho} \cdot h^u[(\nabla_u g)(u)] = 0 \ , \tag{96}$$

or in the form

$$\overline{\rho} \cdot a - \overline{g}(Krp) - \overline{f} \cdot u - h^u[p \cdot \delta Kr - \overline{\rho} \cdot (\nabla_u g)(u)] = 0 \quad , \tag{97}$$

where

$$\overline{f} := \frac{1}{e} \cdot [\overline{\rho} \cdot g(u, a) - g(u, \overline{g}(Krp))] \quad , \tag{98}$$

$$h^u[p \cdot \delta Kr - \overline{\rho} \cdot (\nabla_u g)(u)] = p \cdot h^u(\delta Kr) - \overline{\rho} \cdot h^u[(\nabla_u g)(u)] \ . \tag{99}$$

The expression

$$\overline{\rho} \cdot a - \overline{f} \cdot u = \overline{\rho} \cdot (a - \frac{\overline{f}}{\overline{\rho}}) = \overline{\rho} \cdot (a - \widetilde{f} \cdot u) \qquad , \quad \widetilde{f} = \frac{\overline{f}}{\overline{\rho}} \quad , \tag{100}$$

could be transformed in an other expression by changing the time's parameter τ, where $u = d/d\tau$ and $\nabla_u u = a$, with a new time's parameter $\overline{\tau} = \overline{\tau}(\tau)$. Then

$$a - \widetilde{f} \cdot u = [\overline{\alpha}(\overline{u\alpha}) - \widetilde{f} \cdot \overline{\alpha}] \cdot \overline{u} + \overline{\alpha}^2 \cdot \overline{a} \quad , \qquad \overline{a} = \nabla_{\overline{u}} \overline{u} \quad . \tag{101}$$

The function $\overline{\alpha}$ (see above) could be chosen in such a way that the condition

$$\overline{u\alpha} - \widetilde{f} = 0 \tag{102}$$

could be fulfilled. Then

$$\overline{\alpha} = \overline{\alpha}_0 \cdot exp[\int \widetilde{f}(\tau) \cdot d\tau] \quad , \qquad \overline{\alpha}_0 = \text{ const.,} \tag{103}$$

and the Navier-Stokes equation could be written in the form

$$\overline{\rho} \cdot \overline{\alpha}^2 \cdot \overline{a} - \overline{g}(Krp) - h^u[p \cdot \delta Kr - \overline{\rho} \cdot (\nabla_u g)(u)] = 0 \quad , \quad \overline{\rho} :\neq 0 \ , \ \overline{\alpha} \neq 0 \ . \tag{104}$$

or in the form

$$\overline{a} - \frac{1}{\overline{\rho}} \cdot \frac{1}{\overline{\alpha}^2} \cdot \overline{g}(Krp) - \frac{1}{\overline{\rho}} \cdot \frac{1}{\overline{\alpha}^2} \cdot h^u[p \cdot \delta Kr - \overline{\rho} \cdot (\nabla_u g)(u)] = 0 \quad . \tag{105}$$

Special case: U_n-and V_n-spaces. $S := C$, $\nabla_\xi g := 0$ for $\forall \xi \in T(M)$. For this type of spaces the Navier-Stokes equation has the form

$$\overline{a} = \frac{1}{\overline{\rho}} \cdot \frac{1}{\overline{\alpha}^2} \cdot \overline{g}(Krp) \tag{106}$$

because of $\delta Kr = 0$ and $\nabla_u g = 0$.

2. The metric $\overline{g}$ is an arbitrary given metric. We can now introduce a conformal to g metric tensor $\widetilde{g}$

$$\widetilde{g} := \overline{\alpha}^2 \cdot g \ . \tag{107}$$

Then the corresponding contravariant metric $\widetilde{\overline{g}}$ will have the form

$$\widetilde{\overline{g}} = \overline{\alpha}^{-2} \cdot \overline{g} \tag{108}$$

because of the relations

$$\begin{aligned}
\widetilde{g}_{\overline{ik}} \cdot \widetilde{g}^{\,kj} &= \overline{\alpha}^2 \cdot g_{\overline{ik}} \cdot \widetilde{g}^{\,kj} = g_i^j \quad , \\
\overline{\alpha}^2 \cdot g_{\overline{ik}} \cdot g^{il} \cdot \widetilde{g}^{\,kj} &= \overline{\alpha}^2 \cdot \widetilde{g}^{\,jl} = g^{jl} \quad . \\
\widetilde{g}^{\,jl} &= \overline{\alpha}^{-2} \cdot g^{jl} \quad .
\end{aligned}$$

With the new introduced (given) conformal to g metric $\widetilde{g}$ the Navier-Stokes equation could be written as

$$\overline{a} = \frac{1}{\overline{\rho}} \cdot \widetilde{\overline{g}}(Krp) \quad , \tag{109}$$

or in a co-ordinate basis as

$$\overline{a}^i = \frac{1}{\overline{\rho}} \cdot \widetilde{g}^{\,ij} \cdot p_{,j} \tag{110}$$

which is exactly the Euler equation in (pseudo) Riemannian spaces with or without torsion.

3. In a 3-dimensional Euclidean space E_3 the Euler equation will have the form

$$\overline{a}^b = \frac{1}{\overline{\rho}} \cdot \widetilde{g}^{\,bc} \cdot p_{,c} \qquad , \qquad a, b, c = 1, 2, 3 \ . \tag{111}$$

If we now chose the metric $\widetilde{g}$ as the flat metric in E_3, i.e. if $\widetilde{g} = \eta = \eta_{ab} \cdot dx^a.dx^b$, $\eta_{ab} = (1, 1, 1)$ then we obtain the classical Euler equation

$$\overline{a}^b = \frac{1}{\overline{\rho}} \cdot \eta^{\,bc} \cdot p_{,c} \quad , \tag{112}$$

$$\overline{a}^b = \overline{u}^b{}_{;c} \cdot \overline{u}^c \quad , \quad \overline{u}^c := \frac{dx^c}{d\overline{\tau}} = \frac{dx^c}{dt} \quad , \quad d\overline{\tau} = dt \ , \tag{113}$$

which could be written in a form, usually used in the classical hydrodynamics

$$\overline{u}^b{}_{,4} + \overline{u}^a \cdot \overline{u}^b{}_{,a} - \frac{1}{\overline{\rho}} \cdot p_{,b} = 0 \quad , \tag{114}$$

$$\overline{u}^b{}_{,4} = \frac{\partial \overline{u}^b}{\partial t} \quad , \quad \overline{u}^b{}_{,a} = \frac{\partial \overline{u}^b}{\partial x^a} \quad .$$

Therefore, the Navier-Stokes equation for a perfect fluid in a 3-dimensional Euclidean space of the classical hydrodynamics is identical with the Euler equation for the perfect fluid.

4. We can also write the Navier-Stokes equation in the form

$$a - \widetilde{f} \cdot u - \frac{1}{\overline{\rho}} \cdot \overline{g}(Krp) - h^u[\frac{p}{\overline{\rho}} \cdot \delta Kr - (\nabla_u g)(u)] = 0 \quad , \tag{115}$$

where

$$\widetilde{f} = \frac{1}{e} \cdot [g(u, a) - \frac{1}{\overline{\rho}} \cdot g(u, \overline{g}(Krp))] \quad ,$$

$$\overline{\rho} = \rho_\theta + \frac{1}{e} \cdot k \cdot p \quad .$$

If we change the parameter (proper time) τ of the velocity vector field $u = d/d\tau$ with a new parameter $\overline{\tau} = \overline{\tau}(\tau)$, where $\tau = \tau(\overline{\tau})$ with

$$\overline{u} = \frac{d}{d\overline{\tau}} = \frac{d\tau}{d\overline{\tau}} \cdot \frac{d}{d\tau} = \frac{1}{\overline{\alpha}} \cdot u \quad ,$$

$$u = \overline{\alpha} \cdot \overline{u} \quad , \quad \overline{\alpha} = \frac{d\overline{\tau}}{d\tau} \ , \quad \overline{\alpha} \in C^r(M) \quad , \quad r \geq 1 \quad ,$$

then

$$a = \nabla_u u = \nabla_{\overline{\alpha}\cdot\overline{u}}(\overline{\alpha}\cdot\overline{u}) = \overline{\alpha}^2\cdot\overline{a} + (\overline{u}\overline{\alpha})\cdot u \quad , \tag{116}$$

$$\overline{a} = \nabla_{\overline{u}}\overline{u} = \frac{1}{\overline{\alpha}^2}\cdot[a - \frac{1}{\overline{\alpha}}\cdot(u\overline{\alpha})\cdot u] \quad , \qquad \overline{\alpha}\neq 0 \quad . \tag{117}$$

If $\overline{a}$ should be collinear to a (this means no change of the direction of the acceleration a if τ is replaced with $\overline{\tau}$) then

$$\overline{a} = \frac{1}{\overline{\alpha}^2}\cdot a \qquad \text{and} \qquad u\overline{\alpha} = 0 \quad . \tag{118}$$

$u\overline{\alpha} = 0$ means that the function $\overline{\alpha}$ should not change along the trajectory with the parameter τ.

5. Let us now again consider the transformation of the expression

$$a - \widetilde{f}\cdot u$$

under the change of the proper time τ.

$$\begin{aligned} a - \widetilde{f}\cdot u &= \overline{\alpha}^2\cdot\overline{a} + (\overline{u}\overline{\alpha})\cdot u - \widetilde{f}\cdot u = \\ &= \overline{\alpha}^2\cdot\overline{a} + [(\overline{u}\overline{\alpha}) - \widetilde{f}]\cdot u = \\ &= \overline{\alpha}^2\cdot\overline{a} + [\frac{1}{\overline{\alpha}}\cdot(u\overline{\alpha}) - \widetilde{f}]\cdot u \quad . \end{aligned} \tag{119}$$

The last term $[\frac{1}{\overline{\alpha}}\cdot(u\overline{\alpha}) - \widetilde{f}]\cdot u$ could vanish under two types of conditions:

(a) Since $\overline{\alpha}$ is until now an arbitrary given function $\overline{\alpha}(\tau(x^k)) = \overline{\alpha}(x^k) \in C^r(M)$, $r\geq 1$, we could specialize its form by the use of the condition

$$u\overline{\alpha} - \overline{\alpha}\cdot\widetilde{f} = 0 \quad , \qquad \overline{\alpha}\neq 0 \ , \tag{120}$$

leading to the equation

$$\frac{u\overline{\alpha}}{\overline{\alpha}} = \widetilde{f} \quad , \qquad u(log\overline{\alpha}) = \widetilde{f} \quad , \qquad \frac{d}{d\tau}(log\overline{\alpha}) = \widetilde{f} \quad , \tag{121}$$

with the solution

$$\overline{\alpha} = \overline{\alpha}_0\cdot exp(\int \widetilde{f}\cdot d\tau) \qquad , \qquad \overline{\alpha}_0 = \text{ const.} \tag{122}$$

(b) If $\overline{a}$ should be collinear to a then $u\overline{\alpha} = 0$ and $a - \widetilde{f}\cdot u = \overline{\alpha}^2\cdot\overline{a}$ if $\widetilde{f} = 0$.

Remark. If a is collinear to u, i.e. $a := k\cdot u$ then

$$(k - \widetilde{f})\cdot u = \overline{\alpha}^2\cdot\overline{a} + [\frac{1}{\overline{\alpha}}\cdot(u\overline{\alpha}) - \widetilde{f}]\cdot u$$

and

$$\overline{\alpha}^2\cdot\overline{a} = [k - \widetilde{f} - \frac{1}{\overline{\alpha}}\cdot(u\overline{\alpha}) + \widetilde{f}]\cdot u = [k - \frac{1}{\overline{\alpha}}\cdot(u\overline{\alpha})]\cdot u$$

will be also collinear to u. Both equations appear as auto-parallel equations for the vector fields u and $\overline{u}$ collinear to each other. The auto-parallel equations could be considered as a very special type of Navier-Stokes' equation. They could be investigated separately from the more general cases when the trajectories (world lines) of the mass elements are not auto-parallel trajectories for the velocities vectors u and $\overline{u}$.

Therefore, two possible conditions could be imposed for removing the term $\widetilde{f} \cdot u$ from the expression of the Navier-Stokes equation by the use of a transformation of the proper time τ

(a) $\overline{\alpha} = \overline{\alpha}_0 \cdot exp(\int \widetilde{f} \cdot d\tau)$, $\overline{\alpha}_0 =$ const.

(b) $u\overline{\alpha} = 0$, $\widetilde{f} = 0$.

The condition $u\overline{\alpha} = 0$ assures the collinearity of a and $\overline{a}$ even if $\widetilde{f} \neq 0$. From physical point of view, condition (b) is more reasonable because of the condition $u\overline{\alpha} = 0$ leading in any case to

$$\overline{a} = \frac{1}{\overline{\alpha}^2} \cdot a \qquad \text{or to} \qquad a = \overline{\alpha}^2 \cdot \overline{a} \quad . \tag{123}$$

The condition $\widetilde{f} = 0$ has its physical consequences. It could be written in the form (for $e :\neq 0$)

$$g(u, a) = \frac{1}{\rho} \cdot g(u, \overline{g}(Krp)) \quad . \tag{124}$$

In a co-ordinate basis

$$\begin{aligned} g_{\overline{i}\overline{j}} \cdot u^i \cdot a^j &= \frac{1}{\rho} \cdot g_{\overline{i}\overline{j}} \cdot u^i \cdot g^{j\overline{k}} \cdot g_k^m \cdot p_{;m} = \\ &= \frac{1}{\rho} \cdot u^{\overline{i}} \cdot p_{,i} \quad . \end{aligned} \tag{125}$$

Special case: U_n- and V_n-spaces: $S = C$, $\nabla_\xi g = 0$ for $\forall \xi \in T(M)$, $e := g(u, u) := e_0 = \text{const.} \neq 0$.

$$g(u, a) = \frac{1}{2} \cdot [ue - (\nabla_u g)(u, u)] = 0 \ , \tag{126}$$

$$u^i \cdot p_i = \frac{dp}{d\tau} = 0 \quad . \tag{127}$$

If $\widetilde{f} = 0$ then the Navier-Stokes equation (when ${}^{\theta}\overline{\pi} := 0$, ${}^{\theta}\overline{s} := 0$, and ${}^{\theta}\overline{S} := 0$) takes the form of the generalized Euler equation in a $(\overline{L}_n, g)$-space

$$a - \frac{1}{\rho} \cdot \overline{g}(Krp) - h^u[\frac{p}{\rho} \cdot \delta Kr - (\nabla_u g)(u)] = 0 \tag{128}$$

or in a co-ordinate basis

$$a^i = \frac{1}{\rho} \cdot g^{i\overline{j}} \cdot p_{,j} + h^{i\overline{j}} \cdot (\frac{p}{\rho} \cdot g_j^k{}_{;k} - g_{jk;l} \cdot u^l \cdot u^{\overline{k}}) \quad . \tag{129}$$

From the previous expression for a (or for a^i), it follows, after contraction with the vector field u, the condition

$$g(u,a)=\frac{1}{\rho}\cdot g(u,\overline{g}(Krp))+[g(u)](h^u)[\frac{p}{\rho}\cdot\delta Kr-(\nabla_u g)(u)] \quad . \tag{130}$$

Because of $[g(u)](h^u)=0$ we obtain

$$g(u,a)=\frac{1}{\rho}\cdot g(u,\overline{g}(Krp)) \; . \tag{131}$$

This means that the condition $\widetilde{f}=0$ is automatically fulfilled if the generalized Euler equation in a $(\overline{L}_n,g)$-space is valid. Therefore, *the condition* $\widetilde{f}=0$ *appears as a necessary and sufficient condition for the existence of the generalized Euler equation in a* $(\overline{L}_n,g)$*-space* when ${}^{\theta}\overline{\pi}:=0$, ${}^{\theta}\overline{s}:=0$, ${}^{\theta}\overline{S}:=0$.

In conclusion, the Euler equation in a $(\overline{L}_n,g)$-space could be found as a special case of the Navier-Stokes equation under the following conditions

$$\begin{aligned} {}^{\theta}\overline{\pi} &= 0, \quad {}^{\theta}\overline{s}=0, \quad {}^{\theta}\overline{S}=0 \quad , && (132)\\ u\overline{\alpha} &= 0\;, \quad \widetilde{f}=0 \quad . && (133)\end{aligned}$$

The last two conditions are related to the form invariance of the acceleration a (within a scalar factor) with respect to the change of the proper time τ of the world lines of the mass elements of a flow. At the same time these conditions appear as sufficient conditions for the existence of the Euler equation as a special case of the Navier-Stokes equation if ${}^{\theta}\overline{\pi}:=0$, ${}^{\theta}\overline{s}:=0$, ${}^{\theta}\overline{S}:=0$.

5.4 Invariance of the projective metric h_u corresponding to the contravariant non-isotropic (non-null) vector field $u=d/d\tau$ under transformation of the time parameter τ

1. If the time parameter τ is transformed by the use of the transformation $\tau=\tau(\overline{\tau}),\overline{\tau}=\overline{\tau}(\tau)$,

$$u=\frac{d}{d\tau}=\frac{d\overline{\tau}}{d\tau}\cdot\frac{d}{d\overline{\tau}}=\overline{\alpha}\cdot\overline{u} \quad , \quad \overline{\alpha}=\frac{d\overline{\tau}}{d\tau} \quad , \quad \overline{u}=\frac{d}{d\overline{\tau}} \quad , \tag{134}$$

then the corresponding to u projective metrics

$$\begin{aligned} h_u &= g-\frac{1}{e}\cdot g(u)\otimes g(u) \quad , \quad e=g(u,u):\neq 0 \quad , \\ h^u &= \overline{g}-\frac{1}{e}\cdot u\otimes u \;, \end{aligned}$$

do not change, i.e. h_u and h^u are form invariant under the transformation the time parameter τ of the vector field u. On this basis we can prove the following propositions:

Proposition 1. The corresponding to a non-isotropic (non-null) contravariant vector field $u = d/d\tau$ projective metrics h_u and h^u are invariant under the transformation of the parameter τ by means of the transformation $\tau = \tau(\overline{\tau})$ and $\overline{\tau} = \overline{\tau}(\tau)$.

Proof. From the explicit form of h_u

$$h_u = g - \frac{1}{e} \cdot g(u) \otimes g(u)$$

and the form of u after the transformation of τ

$$u = \overline{\alpha} \cdot \overline{u} \quad , \quad \tau = \tau(\overline{\tau}) \quad , \qquad \overline{\alpha} :\neq 0 \ ,$$

it follows that

$$\begin{aligned} h_u &= g - \frac{1}{g(\overline{\alpha} \cdot \overline{u}, \overline{\alpha} \cdot \overline{u})} \cdot g(\overline{\alpha} \cdot \overline{u}) \otimes g(\overline{\alpha} \cdot \overline{u}) = \\ &= g - \frac{1}{\overline{\alpha}^2} \cdot \frac{1}{g(\overline{u}, \overline{u})} \cdot \overline{\alpha}^2 \cdot g(\overline{u}) \otimes g(\overline{u}) = \\ &= g - \frac{1}{g(\overline{u}, \overline{u})} \cdot g(\overline{u}) \otimes g(\overline{u}) = h_{\overline{u}} \quad . \end{aligned} \tag{135}$$

Therefore, $h_u = h_{\overline{u}}$, where $e = g(u, u) = g(\overline{\alpha} \cdot \overline{u}, \overline{\alpha} \cdot \overline{u}) = \overline{\alpha}^2 \cdot g(\overline{u}, \overline{u}) = \overline{\alpha}^2 \cdot \overline{e}$, $g(u) = g(\overline{\alpha} \cdot \overline{u}) = \overline{\alpha} \cdot g(\overline{u})$.

From the explicit form of h^u

$$h^u = \overline{g} - \frac{1}{e} \cdot u \otimes u$$

and the form of u under the transformation of $\tau = \tau(\overline{\tau})$, $u = \overline{\alpha} \cdot \overline{u}$, $\overline{\alpha} :\neq 0$, it follows that

$$\begin{aligned} h^u &= \overline{g} - \frac{1}{g(\overline{\alpha} \cdot \overline{u}, \overline{\alpha} \cdot \overline{u})} \cdot \overline{\alpha} \cdot \overline{u} \otimes \overline{\alpha} \cdot \overline{u} = \\ &= \overline{g} - \frac{1}{g(\overline{u}, \overline{u})} \cdot \overline{u} \otimes \overline{u} = h^{\overline{u}} \quad . \end{aligned} \tag{136}$$

Therefore, $h^u = h^{\overline{u}}$. By that $e = \overline{\alpha}^2 \cdot \overline{e}$, $\overline{e} := g(\overline{u}, \overline{u})$.

Remark. Since the vector field $\overline{u} = (1/\overline{\alpha}) \cdot u$ is collinear to the vector field u it is obvious that the corresponding to u and $\overline{u}$ projective metrics should not change when u change to $\overline{u}$. Nevertheless this conclusion should be proved.

6 Perfect fluids

6.1 Introduction

1. A fluid for which its thermodynamic properties and its viscosity are considered as not important and could be neglected is called a perfect fluid. This

means that the motion of a perfect fluid does not depend on heating and viscous processes [11]. The model of a perfect fluid appears as the simplest model of a fluid. A perfect fluid has no stress tensor (the stress tensor is equal to zero) and no orthogonal to its motion (its velocity) conductive energy flux density and conductive momentum density [12].

A perfect fluid is usually defined and characterized by two different ways:

(a) By means of the Euler-Lagrange equations as equations for a fluid in static or stationary state (Euler's equation, Bernuli's equation).

(b) By means of energy-momentum tensors with vanishing stress tensor, conductive momentum density, and conductive energy flux density. In the relativistic continuous media mechanics the second way is preferred. The reason for that is that the Euler-Lagrange equations appear in general as sufficient but not as necessary conditions for the existence of an energy-momentum tensor of a given type.

Let us now consider the characteristics of a perfect fluid more closely.

Definition. A perfect fluid is a continuous media with an energy-momentum tensor G of the type

$$G := (\rho_G + \frac{1}{e} \cdot k \cdot L) \cdot u \otimes g(u) - L \cdot Kr \quad , \tag{137}$$

where

$$\begin{aligned} \rho_G &= \frac{1}{e^2} \cdot [g(u)](G)(u) \ , \\ k &= \frac{1}{e} \cdot [g(u)](Kr)(u) \quad , \\ Kr &= g^i_j \cdot \partial_i \otimes dx^j \quad , \qquad e = g(u,u) :\neq 0 \quad . \end{aligned}$$

The scalar invariant L is a given Lagrangian invariant, interpreted in the case of fluid mechanics as the pressure p of a fluid, i.e.

$$L := p \quad . \tag{138}$$

Proposition 2. The necessary and sufficient conditions for the existence of a perfect fluid are the conditions:

$$\begin{aligned} {}^G\overline{\pi} &= {}^k\pi = 0 \quad , \\ {}^G\overline{s} &= {}^k s = 0 \quad , \\ {}^G\overline{S} &= {}^k S = 0 \quad . \end{aligned} \tag{139}$$

Proof: 1. Necessity. From the general representation of an energy-momentum tensor G in the form [5], [4]

$$\begin{aligned} G \equiv & (\rho_G + \frac{1}{e} \cdot k \cdot L) \cdot u \otimes g(u) - L \cdot Kr \ + \\ & + u \otimes g({}^k\pi) + {}^k s \otimes g(u) + ({}^k S) g \quad , \end{aligned}$$

it follows that the condition

$$u \otimes g({}^k\pi) + {}^k s \otimes g(u) + ({}^k S)g = 0 \quad , \tag{140}$$

appears as a necessary condition for the existence of a perfect fluid (in accordance to the definition for a perfect fluid).

After contraction of the last relation:

(a) with $g(u)$ from the left side, we obtain

$$e \cdot {}^k\pi = 0 \quad , \quad {}^k\pi = 0 \quad \text{for} \quad e \neq 0 \quad ,$$

because of $g(u, {}^k s) = 0$ and $g(u, ({}^k S)g) = 0$.

(b) with u from the right side, it follows that

$$e \cdot {}^k s = 0 \quad , \quad {}^k s = 0 \quad \text{for} \quad e \neq 0 \quad ,$$

because of $g({}^k\pi, u) = 0$, $({}^k S)g(u) = 0$.

For ${}^k\pi = 0$ and ${}^k s = 0$, it follows that ${}^k S = 0$. Therefore, the necessary condition is equivalent to the conditions (140).

2. Sufficiency. If ${}^k\pi = 0$, ${}^k s = 0$, and ${}^k S = 0$ then

$$u \otimes g({}^k\pi) + {}^k s \otimes g(u) + ({}^k S)g = 0$$

and the energy-momentum tensor G takes the form

$$G = (\rho_G + \frac{1}{e} \cdot k \cdot L) \cdot u \otimes g(u) - L \cdot Kr \quad .$$

The energy-momentum tensor G could also be written in the form

$$G = {}^k G - L \cdot Kr \quad . \tag{141}$$

The tensor ${}^k G$ is called viscous (viscosity) energy-momentum tensor and ${}^k S$ is called stress tensor.

A comparison of the last expression with the expression for G of a perfect fluid leads to the form of ${}^k G$ as

$${}^k G = (\rho_G + \frac{1}{e} \cdot k \cdot L) \cdot u \otimes g(u) \quad . \tag{142}$$

After contraction from the left side with $g(u)$ and from the right side with u, we obtain

$$\begin{aligned} [g(u)]({}^k G)(u) &= (\rho_G + \frac{1}{e} \cdot k \cdot L) \cdot g(u,u) \cdot g(u,u) = \\ &= e^2 \cdot (\rho_G + \frac{1}{e} \cdot k \cdot L) = e^2 \cdot \rho_{{}^k G} \quad , \end{aligned}$$

$$[g(u)]({}^k G)(u) = \rho_G \cdot e^2 + k \cdot e \cdot L \quad ,$$

$$\rho_{^kG} = \frac{1}{e^2} \cdot [g(u)](^kG)(u) = \rho_G + \frac{1}{e} \cdot k \cdot L \quad , \tag{143}$$

$$L = \frac{e}{k} \cdot (\rho_{^kG} - \rho_G) \quad . \tag{144}$$

Therefore, every Lagrangian invariant L obeying the above condition could be used for description a perfect fluid.

If we interpret the Lagrangian invariant L as the pressure p, i.e. if $L = p$, in a dynamic system the condition for L appears as a state equation for the pressure p of a perfect fluid

$$p = \frac{e}{k} \cdot (\rho_{^kG} - \rho_G) = \frac{1}{k \cdot e} \cdot [g(u)](^kG)(u) - \frac{1}{k} \cdot \rho_G \cdot e \quad . \tag{145}$$

2. The energy-momentum tensor G for a perfect fluid could also be written in the form

$$\begin{aligned} G &= (\rho_G + \frac{1}{e} \cdot k \cdot L) \cdot u \otimes g(u) - L \cdot Kr = \\ &= \rho_G \cdot u \otimes g(u) + \frac{1}{e} \cdot L \cdot \{\frac{1}{e} \cdot [g(u)](Kr)(u) \cdot u \otimes g(u) - Kr\} \end{aligned} \tag{146}$$

because of $k = (1/e) \cdot [g(u)](Kr)(u)$.

Therefore,

$$\begin{aligned} G &= \rho_G \cdot u \otimes g(u) - \frac{1}{e} \cdot L \cdot \{Kr - \frac{1}{e} \cdot [g(u)](Kr)(u) \cdot u \otimes g(u)\} \quad , \\ (G)\overline{g} &= \rho_G \cdot u \otimes u - \frac{1}{e} \cdot L \cdot \{(Kr)\overline{g} - \frac{1}{e} \cdot [g(u)](Kr)(u) \cdot u \otimes u\} \quad . \end{aligned} \tag{147}$$

Special case: (L_n, g)-space: $S := C,\ k = 1$.

$$(G)\overline{g} = (\rho_G + \frac{1}{e} \cdot L) \cdot u \otimes u - L \cdot \overline{g} \tag{148}$$

because of

$$(Kr)\overline{g} = g^i_j \cdot g^{jk} \cdot \partial_i \otimes \partial_k = g^{ik} \cdot \partial_i \otimes \partial_k = \overline{g} \, . \tag{149}$$

The tensor $(G)\overline{g}$ could be written in the form

$$\begin{aligned} (G)\overline{g} &= (\rho_G + \frac{1}{e} \cdot L) \cdot u \otimes u - L \cdot \overline{g} = \\ &= \rho_G \cdot u \otimes u - L \cdot (\overline{g} - \frac{1}{e} \cdot u \otimes u) = \\ &= \rho_G \cdot u \otimes u - L \cdot h^u \quad , \\ h^u &= \overline{g} - \frac{1}{e} \cdot u \otimes u \quad . \end{aligned} \tag{150}$$

Therefore,

$$g[(G)\overline{g}] = g_{\overline{ij}} \cdot G^{ij} = \rho_G \cdot g[u \otimes u] - L \cdot g[h^u] \quad .$$

Since

$$g[u \otimes u] = g(u, u) = e \quad , \qquad g[h^u] = n - 1 \quad , \tag{151}$$

we have for $g[(G)\overline{g}]$

$$g[(G)\overline{g}] = \rho_G \cdot e - (n-1) \cdot L \tag{152}$$

and

$$L = \frac{1}{n-1} \cdot (\rho_G \cdot e - g[(G)\overline{g}]) \quad . \tag{153}$$

If $L = p$, i.e. if L is interpreted as the pressure p then p could be expressed in the form

$$p = \frac{1}{n-1} \cdot (\rho_G \cdot e - g[(G)\overline{g}]) \quad . \tag{154}$$

For an energy-momentum tensor G with

$$g[(G)\overline{g}] = 0 \tag{155}$$

we have

$$p = \frac{1}{n-1} \cdot \rho_G \cdot e \quad . \tag{156}$$

Therefore, if a perfect fluid is described in a (L_n, g)-space by means of the corresponding for this fluid energy-momentum tensor G, obeying the additional condition (having the additional property)

$$g[(G)\overline{g}] = g_{ij} \cdot G^{\overline{ij}} = g_{\overline{ij}} \cdot G^{ij} = g_{ij} \cdot G^{ij} = 0 \tag{157}$$

then the pressure p of the perfect fluid is proportional to the rest mass density ρ_G of the corresponding energy-momentum tensor G.

Special case: (L_n, g)-spaces, $dimM := n := 4$, $e := e_0 :=$ const. $:\neq 0$, $g[(G)\overline{g}] = 0$.

$$p = \frac{1}{3} \cdot \rho_G \cdot e \quad . \tag{158}$$

If the pressure p and the energy-momentum tensor G are given, we could find out the value $l_u = \mid u \mid = \mid g(u, u) \mid^{1/2}$ of the velocity vector of the perfect fluid using the form of $e = \pm l_u^2$

$$e = \pm l_u^2 = \frac{1}{\rho_G} \cdot \{(n-1) \cdot p + g[(G)\overline{g}]\} \quad . \tag{159}$$

Remark. The sign before l_u^2 is determined by the signature Sgn g of the metric tensor g. If Sgn $g > 0$ then $e = -l_u^2$. If Sgn $g < 0$ then $e = +l_u^2$.

Special case: (L_n, g)-spaces, $g[(G)\overline{g}] := 0$.

$$\pm l_u^2 = \frac{1}{\rho_G} \cdot (n-1) \cdot p \quad , \tag{160}$$

$$l_u = \sqrt{(n-1) \cdot \mid \frac{p}{\rho_G} \mid} \quad . \tag{161}$$

For $p \geq 0$, $\rho_G > 0$, we have

$$l_u = \sqrt{(n-1) \cdot \frac{p}{\rho_G}} \quad . \tag{162}$$

Special case: (L_n, g)-spaces, $g[(G)\overline{g}] := 0$, $n := 4$, $p \geq 0$, $\rho_G > 0$.

$$l_u = \sqrt{3} \cdot \sqrt{\frac{p}{\rho_G}} \quad . \tag{163}$$

For a $(\overline{L}_n, g)$-space, since $g[G(\overline{g})] = g[(G)\overline{g}]$, we obtain for a perfect fluid, by the use of the explicit form of G, the relations:

$$g[(G)\overline{g}] = (\rho_G + \frac{1}{e} \cdot k \cdot L) \cdot g[u \otimes u] - p \cdot g[(Kr)\overline{g}] \quad .$$

Since

$$\begin{aligned} g[u \otimes u] &= g(u, u) = e \quad , \\ g[(Kr)\overline{g}] &= g_{\overline{i}\overline{k}} \cdot g^i_j \cdot g^{\overline{j}k} = g_{\overline{j}\overline{k}} \cdot g^{\overline{j}k} = g_{\overline{j}\overline{k}} \cdot g^{mk} \cdot f^j{}_m = \\ &= g^m_j \cdot f^j{}_m = f^m{}_m = \overline{f} \quad , \end{aligned}$$

it follows for $g[(G)\overline{g}]$

$$\begin{aligned} g[(G)\overline{g}] &= \rho_G \cdot e + k \cdot p - p \cdot \overline{f} = \\ &= \rho_G \cdot e - (\overline{f} - k) \cdot p \quad . \end{aligned} \tag{164}$$

Then

$$p = \frac{1}{\overline{f} - k} \cdot (\rho_G \cdot e - g[(G)\overline{g}]) \quad , \tag{165}$$

$$e = \frac{1}{\rho_G} \cdot \{(\overline{f} - k) \cdot p + g[(G)\overline{g}]\} \quad . \tag{166}$$

For (L_n, g)-spaces $k = 1$ and $\overline{f} = n$ ($\dim M = n$).

7 Lagrangian formalism for a perfect fluid in special types of $(\overline{L}_n, g)$-spaces and in (L_n, g)-spaces

Let a Lagrangian density of the type

$$\mathbf{L} = \sqrt{-d_g} \cdot p(g_{ij},\, \rho,\, \rho_{;i} \equiv \rho_{,i},\, u^i) \tag{167}$$

with

$$p := p_0 + \overline{p}_1(\rho) + a_0 \cdot \rho \cdot e + \overline{p}_2(u\rho) \tag{168}$$

be given, where

$$\begin{aligned} \overline{p}_2(u\rho) &= \overline{p}_2(u^k \rho_{,k}) \quad , \qquad e = g(u,u) :\neq 0 \ , \\ p_0 &= \text{const.} \ , \qquad a_0 = \text{const.} \ , \\ \rho &= \rho(x^k) \in \otimes^0{}_0(M) \quad . \end{aligned} \tag{169}$$

On the basis of the method of Lagrangians with covariant derivatives (MLCD) the whole scheme of the Lagrangian theory corresponding to the pressure p as Lagrangian invariant could be found in its explicit form.

7.1 Euler-Lagrange's equations

7.1.1 Euler-Lagrange's equation for the invariant function p

For the invariant function ρ the Euler-Lagrange equation is

$$\frac{\partial p}{\partial \rho} - (\frac{\partial p}{\partial \rho_{,i}})_{;i} + q_i \cdot \frac{\partial p}{\partial \rho_{,i}} = 0 \quad , \tag{170}$$

where

$$q = T_{ki}{}^{k} - \frac{1}{2} \cdot g^{\overline{kl}} \cdot g_{kl;i} + g^l_k \cdot g^k_{l;i} \quad .$$

By the use of the relations

$$\begin{aligned} \frac{\partial p}{\partial \rho} &= \frac{\partial \overline{p}_1}{\partial \rho} + a_0 \cdot e \quad , \\ \frac{\partial p}{\partial \rho_{,i}} &= \frac{\partial \overline{p}_2(u^k \cdot \rho_{,k})}{\partial \rho_{,i}} = \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \frac{\partial}{\partial \rho_{;i}} (u^k \cdot \rho_k) = \\ &= \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot u^k \cdot g^i_k = \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot u^i \quad , \end{aligned}$$

$$\begin{aligned} &\frac{\partial \overline{p}_1}{\partial \rho} + a_0 \cdot e - (\frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot u^i)_{;i} + \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot q_i \cdot u^i = \\ &= \frac{\partial \overline{p}_1}{\partial \rho} + a_0 \cdot e - [(\frac{\partial \overline{p}_2}{\partial (u\rho)})_{;i} - \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot q_i] \cdot u^i - \\ &\qquad - \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot u^i{}_{;i} = 0 \quad , \end{aligned}$$

the Euler-Lagrange equation for ρ follows in the form

$$\frac{\partial \overline{p}_1}{\partial \rho} + a_0 \cdot e - [(\frac{\partial \overline{p}_2}{\partial (u\rho)})_{;i} - \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot q_i] \cdot u^i - \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot u^i{}_{;i} = 0 \quad . \tag{171}$$

Special case: $\overline{p}_2 := b_0 \cdot up = b_0 \cdot u^i \cdot p_{,i}, \ b_0 = \text{const.} \neq 0$.

$$\frac{\partial \overline{p}_2}{\partial (u\rho)} = b_0 \ ,$$

$$\frac{\partial \overline{p}_1}{\partial \rho} + a_0 \cdot e + b_0 \cdot q_i \cdot u^i - b_0 \cdot u^i{}_{;i} = 0 \ . \tag{172}$$

7.1.2 Euler-Lagrange's equations for the velocity vector u

The Euler-Lagrange equations (ELEs) for the vector field u could be found in the form

$$\frac{\partial p}{\partial u^i} = a_0 \cdot \rho \cdot \frac{\partial e}{\partial u^i} + \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \frac{\partial (u\rho)}{\partial u^i} = 0 \quad . \tag{173}$$

Since

$$\frac{\partial e}{\partial u^i} = 2 \cdot g_{\overline{i}\overline{k}} \cdot u^k \quad ,$$

$$\frac{\partial (u\rho)}{\partial u^i} = \rho_{,i} \quad ,$$

$$\frac{\partial p}{\partial u^i} = 2 \cdot a_0 \cdot \rho \cdot g_{\overline{i}\overline{k}} \cdot u^k + \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,i} = 0 \quad ,$$

the ELEs for u follow in the form

$$2 \cdot a_0 \cdot \rho \cdot g_{\overline{i}\overline{k}} \cdot u^k + \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,i} = 0 \quad . \tag{174}$$

Special case: $\overline{p}_2 := b_0 \cdot up = b_0 \cdot u^i \cdot p_{,i}$, $b_0 = \text{const.} \neq 0$.

$$2 \cdot a_0 \cdot \rho \cdot g_{\overline{i}\overline{k}} \cdot u^k + b_0 \cdot \rho_{,i} = 0 \quad ,$$

$$\rho_{,i} = -2 \cdot \frac{a_0}{b_0} \cdot \rho \cdot g_{\overline{i}\overline{k}} \cdot u^k \quad . \tag{175}$$

7.1.3 Euler-Lagrange's equations for the metric tensor g

The ELEs for g could be written in the form

$$\frac{\partial p}{\partial g_{kl}} + \frac{1}{2} \cdot p \cdot g^{\overline{k}\overline{l}} = 0 \quad . \tag{176}$$

From the relations

$$\begin{aligned}
\frac{\partial p}{\partial g_{kl}} &= a_0 \cdot \rho \cdot \frac{\partial e}{\partial g_{kl}} = a_0 \cdot \rho \cdot \frac{\partial}{\partial g_{kl}} (g_{ij} \cdot u^{\overline{i}} \cdot u^{\overline{j}}) \\
&= a_0 \cdot \rho \cdot u^{\overline{i}} \cdot u^{\overline{j}} \cdot \frac{\partial g_{ij}}{\partial g_{kl}} \quad , \\
\frac{\partial g_{ij}}{\partial g_{kl}} &= \frac{1}{2} \frac{\partial}{\partial g_{kl}} (g_{ij} + g_{ji}) = \frac{1}{2} \cdot (g_i^k \cdot g_j^l + g_j^k \cdot g_i^l) \quad , \\
\frac{\partial p}{\partial g_{kl}} &= a_0 \cdot \rho \cdot u^{\overline{k}} \cdot u^{\overline{l}} \quad ,
\end{aligned}$$

we obtain

$$\frac{\partial p}{\partial g_{kl}} + \frac{1}{2} \cdot p \cdot g^{\overline{k}\overline{l}} = a_0 \cdot \rho \cdot u^{\overline{k}} \cdot u^{\overline{l}} + \frac{1}{2} \cdot p \cdot g^{\overline{k}\overline{l}} = 0 \quad ,$$

$$g^{kl} = -2 \cdot a_0 \cdot \frac{\rho}{p} \cdot u^k \cdot u^l \quad , \qquad p :\neq 0 \quad . \tag{177}$$

7.1.4 Corollaries from the Euler-Lagrange equations for the metric tensor g

From

$$a_0 \cdot \rho \cdot u^{\overline{k}} \cdot u^{\overline{l}} + \frac{1}{2} \cdot p \cdot g^{\overline{kl}} = 0$$

after contraction with $g_{\overline{kl}}$ we obtain

$$a_0 \cdot \rho \cdot e + \frac{1}{2} \cdot n \cdot p = 0 : \qquad p = -\frac{2}{n} \cdot a_0 \cdot \rho \cdot e \quad . \tag{178}$$

On the other side,

$$p = p_0 + \overline{p}_1(\rho) + a_0 \cdot \rho \cdot e + \overline{p}_2(u\rho) \ .$$

From both the expression for p, the condition

$$(1 + \frac{2}{n}) \cdot a_0 \cdot \rho \cdot e + \overline{p}_1(\rho) + \overline{p}_2(u\rho) = -p_0 = \text{ const.}, \quad n > 0 \ ,$$

follows. The metric tensor components g^{kl} could be represented by means of the components h^{kl} of the projective metric h^u as

$$g^{kl} = h^{kl} + \frac{1}{e} \cdot u^k \cdot u^l$$

and on the other side as

$$g^{kl} = -2 \cdot a_0 \cdot \frac{\rho}{p} \cdot u^k \cdot u^l \quad .$$

From the last two relations, it follows the relation

$$h^{kl} = -(\frac{1}{e} + 2 \cdot a_0 \cdot \frac{\rho}{p}) \cdot u^k \cdot u^l$$

which contradicts to the property of h^u to be orthogonal to the vector field u, i.e. the explicit form of h^{kl} contradicts to the properties of h^u

$$h^u[g(u)] = 0 \quad , \quad h^{ij} \cdot g_{\overline{jk}} \cdot u^k = 0 \ ,$$

because of

$$\begin{aligned} h^{kl} \cdot g_{\overline{lm}} \cdot u^m &= -(\frac{1}{e} + 2 \cdot a_0 \cdot \frac{\rho}{p}) \cdot g_{\overline{lm}} \cdot u^m \cdot u^k \cdot u^l = \\ &= -(\frac{1}{e} + 2 \cdot a_0 \cdot \frac{\rho}{p}) \cdot e \cdot u^k \neq 0 \quad , \\ e &\neq 0 \ , \quad \frac{1}{e} + 2 \cdot a_0 \cdot \frac{\rho}{p} \neq 0 \quad , \end{aligned} \tag{179}$$

Remark. If

$$\frac{1}{e} + 2 \cdot a_0 \cdot \frac{\rho}{p} = 0$$

then

$$\frac{\rho}{p} = -\frac{1}{2 \cdot a_0 \cdot e} \qquad , \qquad p = -2 \cdot a_0 \cdot e \cdot \rho \quad , \quad h^{kl} = 0 \ .$$

The expression (179) means that a projective metric h^u, orthogonal to u, cannot exist if the ELEs for g are fulfilled. Therefore, we cannot consider the sub space, orthogonal to the vector field u, if the ELEs for g are valid. From this point of view, the metric tensor field g could not be considered as dynamic field variable. It should be assumed to be given as a non-dynamic field variable.

7.2 Energy-momentum tensors

7.2.1 Generalized canonical energy-momentum tensor

$$\overline{\theta}_i{}^j = \frac{\partial p}{\partial \rho_{,j}} \cdot \rho_{,i} - p \cdot g_i^j \ . \tag{180}$$

Since

$$\frac{\partial p}{\partial \rho_{,j}} = \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot u^j \ ,$$

we have

$$\overline{\theta}_i{}^j = \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,i} \cdot u^j - p \cdot g_i^j \ . \tag{181}$$

Special case: $\overline{p}_2 := b_0 \cdot up = b_0 \cdot u^i \cdot p_{,i}$, $b_0 = \text{const.} \neq 0$.

$$\overline{\theta}_i{}^j = b_0 \cdot \rho_{,i} \cdot u^j - p \cdot g_i^j \ . \tag{182}$$

If the ELEs for the vector field u are fulfilled then

$$\overline{\theta}_i{}^j = -2 \cdot a_0 \cdot \rho \cdot g_{\overline{ik}} \cdot u^k \cdot u^j - p \cdot g_i^j \ . \tag{183}$$

7.2.2 Symmetric energy-momentum tensor of Belinfante

The symmetric energy-momentum tensor of Belinfante has the simple form

$$_sT_i{}^j = -p \cdot g_i^j \ . \tag{184}$$

7.2.3 Variational energy-momentum tensor of Euler-Lagrange

The variational energy-momentum tensor of Euler-Lagrange has the form

$$\overline{Q}_i{}^j = {}_v\overline{Q}_i{}^j + {}_g\overline{Q}_i{}^j \quad , \tag{185}$$

$$\begin{aligned} {}_v\overline{Q}_i{}^j &= g_k^j \cdot g_i^l \cdot \frac{\delta p}{\delta u^l} \cdot u^k = \frac{\partial p}{\partial u^i} \cdot u^j = \\ &= 2 \cdot a_0 \cdot \rho \cdot g_{\overline{ik}} \cdot u^k \cdot u^j + \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,i} \cdot u^j \quad , \end{aligned} \tag{186}$$

$$
\begin{aligned}
{}_g\overline{Q}_i{}^j &= -2 \cdot \frac{\delta p}{\delta g_{\underline{j}k}} \cdot g_{\bar{i}k} = -2 \cdot \frac{\partial p}{\partial g_{\underline{j}k}} \cdot g_{\bar{i}k} = \\
&= -2 \cdot a_0 \cdot \rho \cdot g_{\overline{ik}} \cdot u^k \cdot u^j \quad .
\end{aligned}
$$

By the use of the above relations, the variational energy-momentum tensor could be written as

$$
\begin{aligned}
\overline{Q}_i{}^j &= {}_v\overline{Q}_i{}^j + {}_g\overline{Q}_i{}^j = \\
&= 2 \cdot a_0 \cdot \rho \cdot g_{\overline{ik}} \cdot u^k \cdot u^j + \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,i} \cdot u^j - \\
-2 \cdot a_0 \cdot \rho \cdot g_{\overline{ik}} \cdot u^k \cdot u^j &= \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,i} \cdot u^j \quad .
\end{aligned} \tag{187}
$$

Special case: $\overline{p}_2 := b_0 \cdot up = b_0 \cdot u^i \cdot p_{,i}, \; b_0 = \text{const.} \neq 0$.

$$
\overline{Q}_i{}^j = b_0 \cdot \rho_{,i} \cdot u^j \quad . \tag{188}
$$

If the ELEs for the vector field u are fulfilled then

$$
\overline{Q}_i{}^j = -2 \cdot a_0 \cdot \rho \cdot g_{\overline{ik}} \cdot u^k \cdot u^j \quad . \tag{189}
$$

7.3 Invariant projections of the energy-momentum tensors

7.3.1 Invariant projections of the generalized energy-momentum tensor θ

(a) Rest mass density ρ_θ.

$$
\begin{aligned}
\rho_\theta &= \frac{1}{e^2} \cdot \overline{\theta}_k{}^i \cdot u_{\bar{i}} \cdot u^{\overline{k}} = \\
&= \frac{1}{e} \cdot \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot u^{\overline{k}} \cdot \rho_{,k} - \frac{1}{e} \cdot k \cdot p \; , \qquad k = \frac{1}{e} \cdot u_{\overline{k}} \cdot u^{\overline{k}} \quad , \\
\rho_\theta &= \frac{1}{e} \cdot [\frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot u^{\overline{k}} \cdot \rho_{,k} - k \cdot p] \quad .
\end{aligned} \tag{190}
$$

Special case: $\overline{p}_2 := b_0 \cdot up = b_0 \cdot u^i \cdot p_{,i}, \; b_0 = \text{const.} \neq 0$.

$$
\rho_\theta = \frac{1}{e} \cdot [b_0 \cdot u^{\overline{k}} \cdot \rho_{,k} - k \cdot p] \quad . \tag{191}
$$

If the ELEs for the vector field u are fulfilled then

$$
\rho_\theta = -\frac{1}{e} \cdot (2 \cdot a_0 \cdot \rho \cdot g_{\overline{kl}} \cdot u^{\overline{k}} \cdot u^l + k \cdot p) \quad . \tag{192}
$$

Special case: (L_n, g)-spaces: $S = C : k = 1$. If the ELEs for u are fulfilled then

$$
\rho_\theta = -(2 \cdot a_0 \cdot \rho + \frac{p}{e}) \; . \tag{193}
$$

(b) Conductive momentum density ${}^{\theta}\overline{\pi}$.

$$
\begin{aligned}
{}^{\theta}\overline{\pi}^{i} &= \frac{1}{e} \cdot {}_{k}\overline{\theta}_{l}{}^{k} \cdot u_{\overline{k}} \cdot h^{\overline{l}i} = \\
&= \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,l} \cdot h^{\overline{l}i} \quad . \qquad (194)
\end{aligned}
$$

Special case: $\overline{p}_2 := b_0 \cdot up = b_0 \cdot u^i \cdot p_{,i}, \ b_0 = \text{const.} \neq 0$.

$$
{}^{\theta}\overline{\pi}^{i} = b_0 \cdot \rho_{,l} \cdot h^{\overline{l}i} \quad . \qquad (195)
$$

Special case: (L_n, g)-spaces: $S = C : k = 1$.

$$
{}^{\theta}\overline{\pi}^{i} = \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,l} \cdot h^{li} \quad . \qquad (196)
$$

If the ELEs for the vector field u are fulfilled then

$$
\begin{aligned}
{}^{\theta}\overline{\pi}^{i} &= -2 \cdot a_0 \cdot \rho \cdot g_{\overline{l}\overline{m}} \cdot u^m \cdot h^{\overline{l}i} = \qquad (197) \\
&= -2 \cdot a_0 \cdot \rho \cdot u_{\overline{l}} \cdot \cdot h^{\overline{l}i} \quad .
\end{aligned}
$$

Special case: (L_n, g)-spaces: $S = C : k = 1$. If the ELEs for u are fulfilled then

$$
{}^{\theta}\overline{\pi}^{i} = -2 \cdot a_0 \cdot \rho \cdot g_{lm} \cdot u^m \cdot h^{li} = 0 \quad . \qquad (198)
$$

Therefore, in a (L_n, g)-space the conductive momentum density ${}^{\theta}\pi$ vanishes if the ELEs for the vector field u are fulfilled.

(c) Conductive energy flux density ${}^{\theta}\overline{s}$.

$$
\begin{aligned}
{}^{\theta}\overline{s}^{i} &= \frac{1}{e} \cdot h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot {}_{k}\overline{\theta}_{l}{}^{k} \cdot u^{\overline{l}} = \qquad (199) \\
&= \frac{1}{e} \cdot \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,l} \cdot u^{\overline{l}} \cdot h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot u^k = 0 \quad .
\end{aligned}
$$

The conductive energy flux density ${}^{\theta}\overline{s}$ is equal to zero for the given Lagrangain density.

(d) Stress tensor ${}^{\theta}\overline{S}$.

$$
\begin{aligned}
{}^{\theta}\overline{S}^{ij} &= h^{ik} \cdot g_{\overline{k}\overline{l}} \cdot {}_{k}\overline{\theta}_{m}{}^{l} \cdot h^{\overline{m}j} = \\
&= \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,m} \cdot u^l \cdot h^{ik} \cdot g_{\overline{k}\overline{l}} \cdot h^{\overline{m}j} = \\
&= \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,m} \cdot h^{\overline{m}j} \cdot h^{ik} \cdot g_{\overline{k}\overline{l}} \cdot u^l = 0 \quad . \qquad (200)
\end{aligned}
$$

The stress tensor ${}^{\theta}\overline{S}$ is equal to zero for the given Lagrangian density.

7.3.2 Invariant projections of the symmetric energy-momentum tensor of Belinfante ${}_sT$

(a) Rest mass density ρ_T.

$$\begin{aligned}\rho_T &= \frac{1}{e^2}\cdot g_{\overline{i}\overline{j}}\cdot u^j\cdot\, {}_sT_k{}^i\cdot u^{\overline{k}} = -\frac{1}{e^2}\cdot p\cdot g_{\overline{i}\overline{j}}\cdot u^j\cdot\cdot u^{\overline{k}}\cdot g^i_k = \\ &= -\frac{1}{e}\cdot k\cdot p \quad .\end{aligned} \tag{201}$$

Special case: (L_n, g)-spaces: $S = C : k = 1$.

$$\rho_T = -\frac{1}{e}\cdot p \quad . \tag{202}$$

(b) Conductive momentum density ${}^T\overline{\pi}$.

$${}^T\overline{\pi}^i = \frac{1}{e}\cdot\, {}_{sk}T_l{}^k\cdot u_{\overline{k}}\cdot h^{\overline{l}i} = 0 \quad , \qquad {}_{sk}T_l{}^k = 0 \quad . \tag{203}$$

(c) Conductive energy flux density ${}^T\overline{s}$.

$${}^T\overline{s}^i = \frac{1}{e}\cdot h^{ij}\cdot g_{\overline{j}\overline{k}}\cdot\, {}_{sk}T_l{}^k\cdot u^{\overline{l}} = 0 \quad , \qquad {}_{sk}T_l{}^k = 0 \quad . \tag{204}$$

(d) Stress tensor ${}^T\overline{S}$.

$${}^T\overline{S}^{\,ij} = h^{ik}\cdot g_{\overline{k}\overline{l}}\cdot\, {}_{sk}T_m{}^l\cdot h^{\overline{m}j} = 0 \quad , \qquad {}_{sk}T_l{}^k = 0 \quad . \tag{205}$$

7.3.3 Invariant projections of the variational energy-momentum tensor of Euler-Lagrange

(a) Rest mass density ρ_Q.

$$\begin{aligned}\rho_Q &= -\frac{1}{e^2}\cdot g_{\overline{i}\overline{j}}\cdot u^j\cdot\overline{Q}_k{}^i\cdot u^{\overline{k}} = -\frac{1}{e^2}\cdot\frac{\partial\overline{p}_2}{\partial(u\rho)}\cdot\rho_{,k}\cdot u^i\cdot u_{\overline{i}}\cdot u^{\overline{k}} = \\ &= -\frac{1}{e}\cdot\frac{\partial\overline{p}_2}{\partial(u\rho)}\cdot\rho_{,k}\cdot u^{\overline{k}} \quad .\end{aligned} \tag{206}$$

Special case: $\overline{p}_2 := b_0\cdot up = b_0\cdot u^i\cdot p_{,i},\ b_0 = \text{const.}\neq 0$.

$$\rho_Q = -\frac{1}{e}\cdot b_0\cdot\rho_{,k}\cdot u^{\overline{k}} \quad . \tag{207}$$

If the ELEs for the vector field u are fulfilled then

$$\begin{aligned}\rho_Q &= -\frac{1}{e}\cdot(-2\cdot a_0\cdot\rho\cdot g_{\overline{k}\overline{l}}\cdot u^l)\cdot u^{\overline{k}} = \\ &= \frac{2}{e}\cdot a_0\cdot\rho\cdot g_{\overline{k}\overline{l}}\cdot u^l\cdot u^{\overline{k}} \ .\end{aligned} \tag{208}$$

Special case: (L_n, g)-spaces: $S = C : k = 1$. If the ELEs for u are fulfilled then

$$\rho_Q = 2 \cdot a_0 \cdot \rho \quad . \tag{209}$$

For $a_0 := \frac{1}{2}$ the rest mass density ρ_Q is identical to the introduced invariant function ρ in the Lagrangian invariant $L = p$.

(b) Conductive momentum density ${}^Q\pi$.

$$\begin{aligned} {}^Q\pi^i &= -\frac{1}{e} \cdot \overline{Q}_l{}^k \cdot u_{\overline{k}} \cdot h^{\overline{l}i} = -\frac{1}{e} \cdot \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,l} \cdot u^k \cdot u_{\overline{k}} \cdot h^{\overline{l}i} = \\ &= -\frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,l} \cdot h^{\overline{l}i} \quad . \end{aligned} \tag{210}$$

Special case: $\overline{p}_2 := b_0 \cdot up = b_0 \cdot u^i \cdot p_{,i}, \ b_0 = \text{const.} \neq 0$.

$${}^Q\pi^i = -b_0 \cdot \rho_{,l} \cdot h^{\overline{l}i} \quad . \tag{211}$$

If the ELEs for the vector field u are fulfilled then

$${}^Q\pi^i = 2 \cdot a_0 \cdot \rho \cdot g_{\overline{l}\overline{m}} \cdot u^m \cdot h^{\overline{l}i} \quad . \tag{212}$$

Special case: (L_n, g)-spaces: $S = C : k = 1$. If the ELEs for u are fulfilled then

$${}^Q\pi^i = 2 \cdot a_0 \cdot \rho \cdot g_{lm} \cdot u^m \cdot h^{li} = 0 \quad . \tag{213}$$

(c) Conductive energy flux density ${}^Q s$.

$$\begin{aligned} {}^Q s^i &= -\frac{1}{e} \cdot h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot \overline{Q}_l{}^k \cdot u^{\overline{l}} = \\ &= -\frac{1}{e} \cdot \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,l} \cdot u^k \cdot h^{ij} \cdot g_{\overline{j}\overline{k}} \cdot u^{\overline{l}} = 0 \quad . \end{aligned} \tag{214}$$

(d) Stress tensor ${}^Q S$.

$$\begin{aligned} {}^Q S^{ij} &= -h^{ik} \cdot g_{\overline{k}\overline{l}} \cdot \overline{Q}_m{}^l \cdot h^{\overline{m}j} = \\ &= -h^{ik} \cdot g_{\overline{k}\overline{l}} \cdot \frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,m} \cdot u^l \cdot h^{\overline{m}j} = \\ &= -\frac{\partial \overline{p}_2}{\partial (u\rho)} \cdot \rho_{,m} \cdot h^{\overline{m}j} \cdot h^{ik} \cdot g_{\overline{k}\overline{l}} \cdot u^l = 0 \quad . \end{aligned} \tag{215}$$

We can summarize the results for the projective quantities if the ELEs for the vector field u are fulfilled in a $(\overline{L}_n, g)$-space and in a (L_n, g)-space in the following table

Quantity Space	ρ_θ	${}^\theta\overline{\pi}$	${}^\theta\overline{s}$	${}^\theta\overline{S}$	ρ_T	${}^T\overline{\pi}$	${}^T\overline{s}$	${}^T\overline{S}$	ρ_Q	${}^Q\pi$	${}^Q s$	${}^Q S$
$(\overline{L}_n, g)$-space	$\neq 0$	$\neq 0$	0	0	$\neq 0$	0	0	0	$\neq 0$	$\neq 0$	0	0
(L_n, g)-space	$\neq 0$	0	0	0	$\neq 0$	0	0	0	$\neq 0$	0	0	0

It is obvious from the table that a perfect fluid described by the use of the given Lagrangian density $\mathbf{L}$ could exist only in a (L_n, g)-space and, in general, it could not exist in a $(\overline{L}_n, g)$-space. In special cases of $(\overline{L}_n, g)$-spaces with conformal contraction operator S with $S(dx^i, \partial_j) = f^i{}_j := \varphi(x^k) \cdot g^i_j$ there are analogous relations as in (L_n, g)-spaces. In a (L_n, g)-space:

$$\theta = (\rho_\theta + \frac{1}{e} \cdot p) \cdot u \otimes g(u) - p \cdot Kr \ , \quad \rho_\theta = -(2 \cdot a_0 \cdot \rho + \frac{p}{e}) \ , \tag{216}$$

$$_sT = -p \cdot Kr \ , \quad \rho_T = -\frac{p}{e} \ , \tag{217}$$

$$Q = -\rho_Q \cdot u \otimes g(u) \ , \quad \rho_Q = 2 \cdot a_0 \cdot \rho \ , \tag{218}$$

$$\theta = -2 \cdot a_0 \cdot \rho \cdot u \otimes g(u) - p \cdot Kr \ , \tag{219}$$

$$_sT = -p \cdot Kr \ , \tag{220}$$

$$Q = -2 \cdot a_0 \cdot \rho \cdot u \otimes g(u) \ . \tag{221}$$

If we chose the arbitrary constant $a_0 \neq 0$ as $a_0 := -\frac{1}{2}$ then

$$\rho_\theta = \rho - \frac{p}{e} \ , \tag{222}$$

$$\rho_T = -\frac{p}{e} \ , \tag{223}$$

$$\rho_Q = -\rho \ . \tag{224}$$

The scalar invariant function ρ is proportional to the rest mass density ρ_Q of the variational energy-momentum tensor of Euler-Lagrange. The rest mass density ρ_T of the symmetric energy-momentum tensor of Belinfante is equal up to a sign to the rest mass density generated by the pressure p of the system. The rest mass density ρ_θ corresponding to the generalized canonical energy-momentum tensor is equal to the difference of the rest mass ρ and $\frac{p}{e}$ (if $a_0 = -\frac{1}{2}$).

The generalized canonical energy-momentum tensor could be considered as the sum of the symmetric energy-momentum tensor of Belinfante and the variational energy-momentum tensor of Euler-Lagrange

$$\theta \equiv {}_sT + Q \ . \tag{225}$$

The last relation represents the second covariant Noether identity for a perfect fluid.

In the further considerations we will assume the existence of a perfect fluid in a (L_n, g)-space or in its special cases.

8 Navier-Stokes' and Euler's equations in (L_n, g)-spaces

8.1 Navier-Stokes' equations

In a (L_n, g)-space ($S = C$), if the Euler-Lagrange equations for the scalar function ρ and for the vector field u are valid, we have the relations for a perfect

fluid in a co-ordinate basis:

$$g^{ik} \cdot \overline{\theta}_k{}^j{}_{;j} = (\rho_\theta + \frac{1}{e} \cdot p) \cdot a^i + $$
$$+[(\rho_\theta + \frac{1}{e} \cdot p)_{,j} \cdot u^j + (\rho_\theta + \frac{1}{e} \cdot p) \cdot u^j{}_{;j}] \cdot u^i -$$
$$-p_{,j} \cdot g^{ij} + (\rho_\theta + \frac{1}{e} \cdot p) \cdot g^{il} \cdot g_{lj;k} \cdot u^j \cdot u^k \quad . \tag{226}$$

Since

$$\rho_\theta = -(2 \cdot a_o \cdot \rho + \frac{1}{e} \cdot p) \quad ,$$

we have further

$$g^{ik} \cdot \overline{\theta}_k{}^j{}_{;j} = -2 \cdot a_o \cdot \rho \cdot a^i -$$
$$-2 \cdot a_0 \cdot (\rho_{,j} \cdot u^j + \rho \cdot u^j{}_{;j}) \cdot u^i - p_{,j} \cdot g^{ij} -$$
$$-2 \cdot a_0 \cdot \rho \cdot g^{il} \cdot g_{lj;k} \cdot u^j \cdot u^k \quad . \tag{227}$$

The Navier-Stokes equation could be written in the forms

$$\text{(a)} \qquad h_u[\overline{g}(\delta\theta)] = 0 \quad , \tag{228}$$

$$\text{(b)} \qquad h_{li} \cdot g^{ik} \cdot \overline{\theta}_k{}^j{}_{;j} = 0 \tag{229}$$

Since (see the above consideration of Navier-Stokes' equation)

$$\overline{\rho} = \rho_\theta + \frac{1}{e} \cdot p = -2 \cdot a_0 \cdot \rho \quad , \qquad k = 1 \quad , \qquad \delta Kr = 0 \quad , \tag{230}$$

the Navier-Stokes equation for a perfect fluid in a (L_n, g)-space could also be written in the forms [see the special case of $(\overline{L}_n, g)$-spaces with ${}^G\overline{\pi} := {}^k\pi = 0$, ${}^G\overline{s} := {}^ks = 0$, ${}^G\overline{S} := {}^kS = 0$.]

$$-2 \cdot a_0 \cdot \rho \ \cdot a - \overline{g}(Krp) + \frac{1}{e} \cdot [2 \cdot a_0 \cdot \rho \cdot g(u,a) + g(u, \overline{g}(Krp))] \cdot u -$$

$$-2 \cdot a_0 \cdot \rho \ \cdot h^u[(\nabla_u g)(u)] = 0 \quad . \tag{231}$$

$$2 \cdot a_0 \cdot \rho \ \cdot a = -\overline{g}(Krp) + \frac{1}{e} \cdot [2 \cdot a_0 \cdot \rho \cdot g(u,a) + g(u, \overline{g}(Krp))] \cdot u -$$
$$-2 \cdot a_0 \cdot \rho \ \cdot h^u[(\nabla_u g)(u)] \quad , \tag{232}$$

$$2 \cdot a_0 \cdot \rho \ \cdot a = -\overline{g}(Krp) - 2 \cdot a_0 \cdot \rho \ \cdot h^u[(\nabla_u g)(u)] + f \cdot u \quad , \tag{233}$$

where

$$f = \frac{1}{e} \cdot [2 \cdot a_0 \cdot \rho \cdot g(u,a) + g(u, \overline{g}(Krp))] \quad . \tag{234}$$

Special case: U_n- and V_n-spaces: $\nabla_\xi g := 0$ for $\forall \xi \in T(M)$.

$$2 \cdot a_0 \cdot \rho \ \cdot a = -\overline{g}(Krp) + f \cdot u \quad . \tag{235}$$

If we apply now the general consideration for the transformation of the proper time τ of the vector field $u = d/d\tau$ we can find the Navier-Stokes equation in a new form by the use of the relations

$$\begin{aligned} a &= \overline{\alpha}^2 \cdot \overline{a} \quad , \qquad \overline{\alpha} = \overline{\alpha}_0 \cdot exp[\widetilde{f} \cdot d\tau] \quad , \qquad \overline{\alpha}_0 = \text{ const.}, \\ \widetilde{f} &= \frac{1}{2 \cdot a_0 \cdot \rho} \cdot f \quad , \end{aligned}$$

$$a = -\frac{1}{2 \cdot a_0 \cdot \rho} \cdot \overline{g}(Krp) - h^u[(\nabla_u g)(u)] + \widetilde{f} \cdot u \quad , \tag{236}$$

$$a - \widetilde{f} \cdot u = \overline{\alpha}^2 \cdot \overline{a} \quad ,$$

$$\overline{\alpha}^2 \cdot \overline{a} = -\frac{1}{2 \cdot a_0 \cdot \rho} \cdot \overline{g}(Krp) - h^u[(\nabla_u g)(u)] \quad . \tag{237}$$

For the new proper time parameter $\overline{\tau}$ the Navier-Stokes equation has the form

$$\overline{a} = -\frac{1}{2 \cdot a_0 \cdot \rho} \cdot \frac{1}{\overline{\alpha}^2} \cdot \overline{g}(Krp) - \frac{1}{\overline{\alpha}^2} \cdot h^u[(\nabla_u g)(u)] \quad . \tag{238}$$

Since the metric tensor field $\overline{g}$ is an arbitrary given contravariant metric, corresponding to the covariant metric tensor g, we can introduce a metric $\widetilde{g} := \overline{\alpha}^2 \cdot g$, conformal to the metric g. Then the corresponding contravariant metric $\widetilde{\overline{g}}$ will have the form

$$\widetilde{\overline{g}} = \frac{1}{\overline{\alpha}^2} \cdot \overline{g} \quad . \tag{239}$$

For the new introduced metric $\widetilde{g}$ the Navier-Stokes equation could be written in the form

$$\overline{a} = -\frac{1}{2 \cdot a_0 \cdot \rho} \cdot \widetilde{\overline{g}}(Krp) - h^u[\frac{1}{\overline{\alpha}^2} \cdot (\nabla_u g)(u)] \quad . \tag{240}$$

On the other side,

$$\begin{aligned} u &= \overline{\alpha} \cdot \overline{u} \quad , \qquad h^u = h^{\overline{u}} \quad , & (241) \\ \nabla_u g &= \nabla_{\overline{\alpha} \cdot \overline{u}} g = \overline{\alpha} \cdot \nabla_{\overline{u}} g \quad , & (242) \\ (\nabla_u g)(u) &= (\nabla_{\overline{\alpha} \cdot \overline{u}} g)(\overline{\alpha} \cdot \overline{u}) = \overline{\alpha}^2 \cdot (\nabla_{\overline{u}} g)(\overline{u}) \quad , & (243) \\ \frac{1}{\overline{\alpha}^2} \cdot (\nabla_u g)(u) &= (\nabla_{\overline{u}} g)(\overline{u}) \quad . & (244) \end{aligned}$$

By the use of the above relations the Navier-Stokes equation takes the form

$$\overline{a} = -\frac{1}{2 \cdot a_0 \cdot \rho} \cdot \widetilde{\overline{g}}(Krp) - h^{\overline{u}}[(\nabla_{\overline{u}} g)(\overline{u})] \quad , \tag{245}$$

which is exactly the generalization of the Euler equation for (L_n, g)-spaces. In U_n- and V_n-spaces $(\nabla_{\overline{u}} g = 0)$ and for $a_0 := \frac{1}{2} \cdot \varepsilon$ with $(\varepsilon = \mp 1)$ the equation takes its standard form

$$\overline{a} = \varepsilon \cdot \frac{1}{\rho} \cdot \widetilde{\overline{g}}(Krp) \tag{246}$$

or in a co-ordinate basis

$$\overline{a}^i = \varepsilon \cdot \frac{1}{\rho} \cdot \widetilde{g}^{ij} \cdot p_{,j} \quad , \qquad \varepsilon = \mp 1 \quad . \tag{247}$$

Remark. The sign of ε is depending on the sign of the pressure p defined by different authors as extrovert (acting out) or introvert (acting into) quantity with respect to a dynamical system.

9 Conclusion

In this paper the Euler equation for a perfect fluid is considered as a special case of Navier-Stokes' equation in spaces with affine connections and metrics. By the use of a special type of a Lagrangian density the Navier-Stokes equations are obtained. It turns out that these equations could be considered as equations for a perfect fluid in spaces with one affine connection and a metric [(L_n, g)-spaces] and only in special cases of spaces with contravariant and covariant affine connections and metrics [$(\overline{L}_n, g)$-spaces].

The Euler equation is obtained by the use of entirely unconstrained Lagrangian formalism represented by the method of Lagrangians with covariant derivatives (MLCD). The additional conditions by this method are related to the transformation properties of the proper time of a mass element (particle) moving in space-time. They are not related to any constraints of the variational principle used for finding out the Euler-Lagrange equations for a perfect fluid. Therefore, we can consider unconstrained variational principles in hydrodynamics but under the condition for more careful use of the notion of proper time (or of time parameter) and its transformation properties. It seams that the notion of proper time (as the notion of time in general) could be much more important in physics than assumed until now.

References

[1] Manoff S., *Invariant projections of energy-momentum tensors for field theories in spaces with affine connection and metric.* J. Math. Phys. **32** (1991) 3, 728-734

[2] Manoff S. and Lazov R., *Invariant projections and covariant divergency of the energy-momentum tensors.* In *Aspects of Complex Analysis, Differential Geometry and Mathematical Physics.* Eds. S. Dimiev, K. Sekigava, (World Scientific, Singapore 1999), pp. 289-314 [Extended version: E-print (1999) gr-qc/99 07 085]

[3] Manoff S., *Geometry and mechanics in different models of space-time: Geometry and kinematics.* (Nova Science Publ., New York, 2002)

[4] Manoff S., *Geometry and mechanics in different models of space-time: Dynamics and applications.* (Nova Science Publ., New York, 2002)

[5] Manoff S., *Lagrangian theory of tensor fields over spaces with contravariant and covariant affine connections and metrics and its applications to Einstein's theory of gravitation in $\overline{V}_4$-spaces.* Acta Appl. Math. **55** (1999) 1, 51-125

[6] Schutz B. F. and Sorkin R., *Variational Aspects of Relativistic Field Theories with Application to Perfect Fluids*. Ann. of Phys. (USA) **107** (1977), 1-43

[7] Manoff S., *Spaces with contravariant and covariant affine connections and metrics.* Physics of elementary particles and atomic nucleus (Physics of Particles and Nuclei) [Russian Edition: **30** (1999) 5, 1211-1269], [English Edition: **30** (1999) 5, 527-549]

[8] Manoff S., *Lagrangian formalism for tensor fields.* In *Topics in Complex Analysis, Differential Geometry and Mathematical Physics*. eds. Dimiev St., Sekigawa K. (World Sci. Publ., Singapore, 1997), pp. 177-218

[9] Manoff S., *Frames of reference in spaces with affine connections and metrics*. Class. Quantum Grav. **18** (2001) 6, 1111-1126. E-print (1999) gr-qc / 99 08 061

[10] Lovelock D., Rund H., *Tensors, Differential Forms, and Variational Principles* (John Wiley & Sons, New York, 1975)

[11] Landau L. D., Lifshitz E. M., *Gidrodinamika* (*Hydrodynamics*) (Nauka, Moscow, 1986) (in Russian)

[12] Loitzjanskii L. G., *Mechanics of fluids and gases*. 6th edition. (Nauka, Moscow, 1987) (in Russian)

[13] Manoff S., *Auto-parallel equation as Euler-Lagrange's equation in spaces with affine connections and metrics.* Gen. Rel. and Grav. **32** (2000) 8, 1559-1582. E-print gr-qc / 00 10 048

[14] Manoff S., *Flows and particles with shear-free and expansion-free velocities in $(\overline{L}_n, g)$- and Weyl's spaces*. Class. Quantum Grav. **19** (2002) 16, 4377-4398. E-print gr-qc / 02 07 060